The Balanced Solution for TU–games

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February 2007

Abstract

The Shapley value of a TU-game distributes the dividend of each coalition in the game equally among the players in the coalition. Given exogenous weights for all players, the corresponding weighted Shapley value distributes dividends proportional to the weights of the players. In the solution defined in this contribution these weights satisfy the property that the corresponding weighted Shapley value of each player is equal to its weight. We refer to our solution as a balanced solution and we show its existence for all monotone TU-games. Finally, we provide a characterization of balanced solutions using a reduced game consistency and discuss some properties of our solution.

JEL Classification: C71

Key words: Balanced solution, Proportionality, Reduced game consistency, Weighted Shapley value,

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1 Introduction

A situation in which a finite set of \( n \) players \( N = \{1, \ldots, n\} \) can generate certain payoffs by cooperation can be described by a *cooperative transferable utility game* (or simply a TU-game), being a pair \((N, v)\) where \( v: 2^N \rightarrow \mathbb{R} \) is a *characteristic function* on \( N \) satisfying \( v(\emptyset) = 0 \). For any coalition \( S \subseteq N \), \( v(S) \) is the *worth* of coalition \( S \), i.e. the members of coalition \( S \) can obtain a total payoff of \( v(S) \) by agreeing to cooperate.

A *payoff vector* \( n \)-player TU-game is an \( n \)-dimensional vector which components are the payoffs that the corresponding players earn in the game. A (single-valued) *solution* for TU-games is a function that assigns a payoff vector to every TU-game. The most known solutions for TU-games is the *Shapley value* (Shapley (1953a)) which distributes the so-called Harsanyi dividends of the game equally among the players in the corresponding coalition. For exogenously given weights \( \omega \in \mathbb{R}^N_{++} \), Shapley (1953b) introduced the *weighted Shapley value* that assigns to the game \((N, v) \in \mathcal{G}\) the payoff vector that distributes the dividend of every coalition (see Harsanyi, 1959) proportional to the weights of players in the coalition. The weight scheme \( \omega \) describes in some way the bargaining ability of the players. Since also the weighted Shapley value itself depicts the power of particular players in the game, it is natural to examine weight vectors for which the weighted Shapley value produces exactly the initial weights.

This construction has an intuitively clear consequence for the 2-person games, where the above mentioned solution distributes the dividend of the grand coalition *proportionally* to the individual worths. This fact makes the weight vectors for which the weighted Shapley value produces exactly the initial weights particularly interesting for situations where proportionality is more natural than equality. It is, for example, standard business practice that the profit of a firm is divided proportionally to the investment. Consider two players facing the following investment opportunity: at the market where the interest rate is 1% for amounts below 500 thousands and 2% for amounts higher or equal to 500 thousands there are two investors with endowments 100 thousands and 400 thousands. In the underlying TU-game the small investor receives 1 thousand, the big investor 4 thousands and the coalition of cooperating investors 10 thousands. It seems to be natural to expect that cooperating investors will agree that both enjoy the same high interest rate (2%) and the profit will be split proportionally (1:4). On contrary, the Shapley value predicts equal split of surplus which gives to the small investor 3.5 thousands (3.5%) and
6.5 thousand (1.625% interest rate) to the big investor. The fact that the small investor should enjoy more than two times higher interest seems at least questionable.

With similar ideas in mind, several authors already proposed ‘proportional’ solutions for particular classes of games (see, e.g. Kalai (1977) or Roth (1979)). In cooperative game theory, Vorob’ev and Lyapunov (1998) define Proper Shapley values based on the above mentioned weighted Shapley value with weights that are endogenously determined so that they satisfy the property that the corresponding weighted Shapley value of each player is equal to its weight. These solutions are well-defined for the class of games with non-negative dividends. For economic applications this is a rather narrow class that is contained in the class of monotone, convex games\(^1\). Another solution that is based on proportionality is the Proportional value introduced by Feldman (1999) and Ortmann (2000) which is based on a modified difference potential (Hart and Mas-Colell (1989)). However, this solution can be applied only to games with positive worths (except for the empty set), such that it excludes many interesting classes of games such as the class of weighted voting games.

Our main aim in this paper is to define a solution that is based on proportionality and is well-defined for all monotone games. Similarly as Vorob’ev and Lyapunov (1998), we consider fixed points of a particular mapping on the payoff simplex. Our mapping coincides with the weighted Shapley value in case all weights are positive, i.e. for weights in the interior of the efficient, non-negative payoff simplex. However, for weights on the boundary of this simplex we split the dividends equally among the players in the corresponding coalition. We refer to weights obtained in this way as *balanced weights* and to the corresponding payoff vector as a *balanced value*. We show existence of balanced values whenever the game is monotone.

Having the existence theorem at hand we provide an axiomatization of the balanced solution by adapting the standardness for two-player games and the HM-reduced game property, defined in characterizing the Shapley value by Hart and Mas-Colell (1988, 1989). As a next step, we show some properties of balanced solutions such as component efficiency, component restriction, the null player property and, in particular, individual rationality. Finally, all proofs are relegated to an appendix.

\(^1\)Monotone, convex games might have negative dividends as illustrated by the 3-person game \((N, v)\) with \(v(S) = |S| - 1\) where \(\Delta(N) = -1\).
2 Preliminaries: TU-Games and solutions

A cooperative game with transferable utility \((N, v)\) is monotone if \(v(S) \leq v(T)\) whenever \(S \subseteq T \subseteq N\). A TU-game \((N, v)\) is superadditive if \(v(S \cup T) \geq v(S) + v(T)\) for any disjoint \(S, T \subseteq N\) (i.e. \(S \cap T = \emptyset\)). We denote the collection of all TU-games by \(G\), the collection of all monotone TU-games by \(G_M\) and the collection of all superadditive TU-games by \(G_A\).

Every game \((N, v) \in G\) can be alternatively represented by its dividends \(\Delta_{N,v}(S)\), \(S \subseteq N, S \neq \emptyset\), given by (see Harsanyi, 1959)

\[
\Delta_{N,v}(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T),
\]

where \(|S|\) denotes the cardinality of \(S\). Dividends may alternatively be defined recursively as \(\Delta_{N,v}(\{i\}) = v(\{i\})\) if \(S = \{i\}\), and \(\Delta_{N,v}(S) = v(S) - \sum_{T \subseteq S, T \neq S} \Delta_{N,v}(T)\) if \(|S| \geq 2\).

A payoff vector \(x \in \mathbb{R}^n\) of a TU-game \((N, v)\) is an \(n\)-dimensional vector giving a payoff \(x_i \in \mathbb{R}\) to any player \(i \in N\). A payoff vector \(x\) for game \((N, v)\) is efficient if it exactly distributes the worth \(v(N)\) of the ‘grand coalition’ \(N\), i.e. if \(\sum_{i \in N} x_i = v(N)\). The set of efficient payoff vectors of \((N, v) \in G\) is denoted by \(X(N, v) = \{x \in \mathbb{R}^n | \sum_{i \in N} x_i = v(N)\}\).

A single-valued solution on a subset \(C\) of \(G\) is a function \(f\) that assigns to every game \((N, v) \in C\) a payoff vector \(f(N, v) \in \mathbb{R}^n\). A single-valued solution \(f\) is efficient on \(C\) if \(f(N, v)\) is an efficient payoff vector for all \((N, v) \in C\). The Shapley value (1953a) is an efficient single-valued solution, obtained by distributing the dividends of every coalition equally among all players in the coalition, i.e. it is the function \(\varphi\) defined by

\[
\varphi_i(N, v) = \frac{1}{|S|} \sum_{S \subseteq N, i \in S} \Delta_{N,v}(S). \tag{1}
\]

A set-valued solution \(F\) on a subset \(C\) of \(G\) assigns a set of payoff vectors \(F(N, v) \subseteq \mathbb{R}^n\) to any game \((N, v) \in G\). A set-valued solution \(F\) is efficient on \(C \subseteq G\) if every payoff vector in \(F(N, v)\) is efficient for all \((N, v) \in C\).

Given positive weight vector \(\omega \in \mathbb{R}_{++}^n\) with weights \(\omega_i > 0, i \in N\), the corresponding weighted Shapley value (Shapley, 1953b) on \(G\) is the function \(\varphi^\omega\) defined by

\[
\varphi_i^\omega(N, v) = \sum_{S \subseteq N, i \in S} \frac{\omega_i}{\omega_j} \Delta_{N,v}(S). \tag{2}
\]

The weighted Shapley value thus distributes the dividends of coalitions proportionally to the exogenously given weights of the players. Clearly, if all weights are equal then it yields the Shapley value: \(\varphi^\omega(N, v) = \varphi(N, v)\) if \(\omega_i = \omega_j\) for all \(i, j \in N\).
In the following, for any vector $x$ and coalition $S$ we denote $x_S = \sum_{i \in S} x_i$ for the sum of the components $x_i, \ i \in S$. The projection of vector $x$ to the set $S$ is denoted as $x|_S$. The restriction of game $(N, v)$ to the set $T \subset N$ is denoted as $(T, v_T)$, where $v_T$ is the characteristic function restricted to the subsets of $T$: $v_T(S) = v(S)$ for all $S \subseteq T$.

3 Balanced Weights and the Balanced Solution

Considering the set of all positive weight vectors, the weighted Shapley value for game $(N, v) \in \mathcal{G}$ can be obtained by the Shapley mapping being the mapping that maps any positive weight vector $x$ into a payoff vector $\varphi^x(N, v)$. This brings attention to the fixed points of the function $\varphi^x(N, v)$. Note that in determining the weighted Shapley value of a game given a positive weight vector, only the ratios of the weights matter, and not the weights itself. Thus, all positive weight vectors $\tilde{\omega}$ such that $\tilde{\omega}_i / \tilde{\omega}_j = \omega_i / \omega_j$ yield the same weighted Shapley value, and thus normalization of the weight vectors is possible. Therefore, for game $(N, v) \in \mathcal{G}$ we could restrict our attention to positive weight vectors in the set $X(N, v)$, i.e. we only consider efficient positive weight vectors, and we consider the function $\varphi(N, v) : X_+(N, v) \to X(N, v)$ with $X_+(N, v) = \{ x \in X(N, v) \mid x_S \neq 0 \text{ for all } S \subseteq N \}$. Since the set $X_+$ is not closed, it is not possible to examine directly the fixed points of $\varphi^x(N, v)$.

To circumvent this caveat, in defining their proper Shapley values, Vorob’ev and Lyapunov (1998) only considered games with non-negative dividends (also known as totally positive games, see Vasil’ev (1978)). For this class of games they could introduce their Proper Shapley values restricting themselves to weights from the compact set $X_+(N, v) = \{ x \in X(N, v) \mid x_i \geq v\{i\}, \ i \in N \}$. Since the intention of our paper is to define a solution for a more general and applicable class containing all monotone games, we define for every $(N, v) \in \mathcal{G}$ the function $h(N, v) : X(N, v) \to X(N, v)$ as

$$h(N, v)_i(x) = \sum_{S \subseteq N : i \in S, x_S \neq 0} \frac{x_i}{x_S} \Delta_{N,v}(S) + \sum_{S \subseteq N : i \in S, x_S = 0} \frac{1}{|S|} \Delta_{N,v}(S). \quad (3)$$

For the sake of brevity, we omit the parameters $(N, v)$ if no confusion is possible, and write only $h$. Now we can look for fixed points of $h$. The set of all fixed points of this function determines a set-valued solution.

**Definition 1** Consider game $(N, v) \in \mathcal{G}$. A vector $x \in X(N, v)$ with $h(N, v)(x) = x$ is called a balanced value. The balanced set of the game $(N, v) \in \mathcal{G}$ is the set of all balanced
values of the game, and is thus given by

\[ B(N, v) = \{ x \in X(N, v) \mid h(N, v)(x) = x \}. \]

We refer to the solution that assigns to every game an element of its balanced set as the balanced solution, and to the components \( x_i, \ i \in N \), of a balanced value of game \((N, v)\) as balanced weights.

In general, fixed points for \( h(N, v) \) need not exist, i.e. \( B(N, v) \) might be empty, as shown by the following example where we discuss all two-player games.

**Example 1** Consider a two-player game \((N, v)\) with \(N = \{1, 2\}\) and characteristic function \( v(\emptyset) = 0, \ v(\{1\}) = \alpha_1, \ v(\{2\}) = \alpha_2 \) and \( v(\{1, 2\}) = \alpha_{12} \). Let \((x_1, x_2)\) be a balanced value for this game. We distinguish the following cases with respect to the worths of the three non-empty coalitions:

1. Suppose that \( \alpha_{12} = 0 \). Efficiency then requires that

\[ x_1 + x_2 = \alpha_{12} = 0. \quad (4) \]

A balanced value \((x_1, x_2)\) for this game then should satisfy:

\[ x_1 = \alpha_1 + \frac{1}{2}(\alpha_{12} - \alpha_1 - \alpha_2) = \frac{1}{2}(\alpha_1 - \alpha_2), \quad (5) \]

and

\[ x_2 = \alpha_2 + \frac{1}{2}(\alpha_{12} - \alpha_1 - \alpha_2) = \frac{1}{2}(\alpha_2 - \alpha_1). \quad (6) \]

Note that for any \( \alpha_1, \alpha_2 \in \mathbb{R} \) the weights \( x_1 \) and \( x_2 \) given by (5) and (6) satisfy the efficiency condition (4), and thus these are balanced weights. So, \( B(N, v) \neq \emptyset \) for \((N, v)\) with \( n = 2 \) and \( v(N) = 0 \). Moreover, there is a unique balanced value.

2. Next, suppose that \( \alpha_{12} \neq 0 \). Then efficiency requires that

\[ x_1 + x_2 = \alpha_{12} \neq 0. \quad (7) \]

A balanced value \((x_1, x_2)\) for this game thus should satisfy:

\[ x_1 = \alpha_1 + \frac{x_1}{x_1 + x_2}(\alpha_{12} - \alpha_1 - \alpha_2), \quad (8) \]

and

\[ x_2 = \alpha_2 + \frac{x_2}{x_1 + x_2}(\alpha_{12} - \alpha_1 - \alpha_2). \quad (9) \]
Substituting the efficiency condition (7) in (8) and (9), respectively, we get
\[(\alpha_1 + \alpha_2)x_i = \alpha_i \alpha_{12} \text{ for } i \in \{1, 2\}.\] (10)

For the case \(\alpha_{12} \neq 0\) we now consider the following two subcases:

If \(\alpha_1 + \alpha_2 \neq 0\), then \((\frac{\alpha_1 \alpha_{12}}{\alpha_1 + \alpha_2}, \frac{\alpha_2 \alpha_{12}}{\alpha_1 + \alpha_2})\) is the unique balanced value.

Otherwise, if \(\alpha_1 + \alpha_2 = 0\), then a balanced value only exists if \(\alpha_i \alpha_{12} = 0\) for \(i \in \{1, 2\}\), i.e. if \(\alpha_1 = \alpha_2 = 0\). In the last case there are multiple balanced values.

In the remaining case with \(\alpha_{12} \neq 0\), \(\alpha_1 + \alpha_2 = 0\) and \(\alpha_1 \alpha_2 \neq 0\) balanced values do not exist. □

If we require monotonicity for two player games, then \(v(\{1\}) = \alpha_1 \geq 0\) and \(v(\{2\}) = \alpha_2 \geq 0\). So, \(v(\{1\}) + v(\{2\}) = 0\) implies that \(v(\{1\}) = v(\{2\}) = 0\), and thus (from the example above) we obtain that \(B(N, v) \neq \emptyset\) for these games. It turns out that this can be generalized for any player set \(N \subseteq N\), i.e. balanced values always exist for any monotone TU-game.

**Theorem 1** If \((N, v)\) is monotone then \(B(N, v) \neq \emptyset\).

The proof of this theorem can be found in Appendix A. It applies Kakutani’s fixed point theorem. However, we cannot directly apply this theorem since the mapping \(h(N, v)\) is not upper hemi-continuous on \(X(N, v)\), as illustrated in the following example.

**Example 2** Consider the monotone game \((N, v) \in G_M\) given by \(N = \{1, 2, 3\}\) with dividends
\[\Delta_{N,v}(S) = \begin{cases} 1 & \text{if } S \in \{\{3\}, \{1, 2\}\} \\ 0 & \text{otherwise} \end{cases}\]
and \(x^\varepsilon = (\varepsilon, 2\varepsilon, 2 - 3\varepsilon)\). Clearly
\[h(N, v)(x^\varepsilon) = (\frac{1}{3}, \frac{2}{3}, 1) \forall \varepsilon > 0\], and, \(h(N, v)(x^\varepsilon) = (\frac{1}{2}, \frac{1}{2}, 1)\) if \(\varepsilon = 0\).

□

In Appendix A we solve this problem for monotone games by restricting \(h(N, v)\) to the simplex of nonnegative efficient payoff vectors, and adding to \(h(N, v)(x)\) it all its limiting points. We then show that Kakutani’s fixed point theorem can be applied to the mapping that assigns to every \(x \in X_0\) the convex envelope of this new set. Finally, we show that the fixed points of this last mapping are also the fixed points of \(h(N, v)\).
Axiomatization

In the previous section we saw that all monotone games have at least one balanced value. In general, a game can have more than one balanced value. In this section we restrict our attention to the class $G_B \supset G_M$ of games with at least one balanced value, and consider single-valued solutions that assign to every such game a balanced value.

Definition 2 A single-valued solution $f$ on $G_B$ is a balanced solution if $f(N, v) \in B(N, v)$ for every $(N, v) \in G_B$.

In Hart and Mas-Colell (1988, 1989) the Shapley value is characterized by standardness for two-player games and the HM-reduced game property. The standardness for two-player games states that in a two-player game each player gets its own worth plus half of the dividend of the two-player coalition: For every $(N, v) \in G$ with $N = \{i, j\}$, $i \neq j$,

$$f_i(N, v) = v(\{i\}) + \frac{1}{2} [v(N) - v(\{i\}) - v(\{j\})].$$ (11)

The HM-reduced game property considers situations in which some players 'leave' the game. The HM-reduced game property states that the payoffs of players that do not leave the game do not change if we consider an appropriately defined reduced game on the set of players remaining in the game. In this reduced game the worth of each coalition equals the worth of the union of this coalition and all players that leave the game minus the payoffs of the leaving players in the corresponding restricted game. For $(N, v) \in G$ with $n \geq 2$, solution $f$ on $G$, and $T \subset N$, the Hart and Mas-Colell reduced game $(T, \overline{v}_f^T)$ is given by

$$\overline{v}_f^T(S) = v(S \cup T^c) - f_{T^c}(S \cup T^c, v_{S \cup T^c})$$ for all $S \subseteq T$, (12)

where $T^c = N \setminus T$ is the coalition of leaving players and $v_{S \cup T^c}$ is the restricted game on $S \cup T^c$ given by $v_{S \cup T^c}(H) = v(H)$ for all $H \subseteq S \cup T^c$, see Section 2. For solutions $f$ and $g$ on $G$ we say that $g$ satisfies the $f$-reduced game property if $g_i(N, v) = g_i(T, \overline{v}_f^T)$ for all $i \in T \subset N$. The HM-reduced game property is a consistency property stating that a solution satisfies its own reduced game property. So, a solution $f$ satisfies the HM-reduced game property if the payoffs of players that do not leave the game is the same in this reduced game and the original game: $f_i(N, v) = f_i(T, \overline{v}_f^T)$ for every $(N, v) \in G$ with $n \geq 2, T \subset N$ and $i \in T$. As shown by Hart and Mas-Colell (1989), a solution $f$ on $G$ satisfies standardness for two-player games and the HM-reduced game property if and
only if it is the Shapley value.

The class of balanced solutions on \( G_B \) is characterized by alternative, proportional versions of the standardness and reduced game properties. First, proportional standardness for two-player games states that the payoffs in two-player games in \( G_B \) are distributed as in Example 1. This implies that whenever the worth of the ‘grand coalition’ is nonzero and the sum of the worths of the singletons is nonzero, the worth \( v(N) \) is distributed proportionally to the individual worths of the players, analogously to Ortmann (2000) and Feldman (1999).

**Axiom 1 (Proportional standardness for two-player games)** For every \( (N, v) \in G_B \) with \( N = \{i, j\}, i \neq j, v(\{i\}) + v(\{j\}) \neq 0 \) it holds that

\[
 f(N, v) = \begin{cases} 
 \left( \frac{v(\{i\})}{v(\{i\}) + v(\{j\})} \right) v(N) + \left( \frac{v(\{j\})}{v(\{i\}) + v(\{j\})} \right) v(N) & \text{if } v(N) \neq 0, \\
 \frac{1}{2}(v(\{i\}) - v(\{j\})) + \frac{1}{2}(v(\{j\}) - v(\{i\})) & \text{if } v(N) = 0
\end{cases}
\]

Note that in the non-trivial case above, where \( v(N) \neq 0, v(\{i\}) + v(\{j\}) \neq 0 \), it holds that \( f_i(N, v) = \frac{v(\{i\})}{v(\{i\}) + v(\{j\})} v(N) = v(\{i\}) + \frac{v(\{i\})}{v(\{i\}) + v(\{j\})} \Delta_{N,v}(N) \). Instead of getting their own worth plus half of the dividend of the grand coalition, as required by the original standardness, the players now get their own worth plus a share in the dividend of the grand coalition proportional to their own worth.

In the proportional reduced game that results after some players leave the game, the dividend of a coalition \( S \) of the remaining players is equal to the dividend of coalition \( S \) in the original game plus a share in the original dividends of all coalitions containing \( S \) and players that leave the game. These shares are determined by the payoffs of the players in the original game. Let \( f \) be a single-valued solution on \( G_B \). For \( (N, v) \in G_B \) with \( n \geq 2 \), \( x = f(N, v) \) and \( T \subseteq N \), the proportional reduced game \( (T, v^x_T) \) is defined by its dividends in the following way. For every \( S \subseteq T \)

\[
\Delta_{T,v^x_T}(S) = \Delta_{N,v}(S) + \sum_{\substack{K \subseteq T \setminus K, K \neq \emptyset \\{x \cup S, K \neq 0}}}^{xS} \Delta_{N,v}(S \cup K)
\]

\[
+ \sum_{\substack{K \subseteq T \setminus K, K \neq \emptyset \\{x \cup S, K = 0}}}^{|S|} \frac{|S|}{|S \cup K|} \Delta_{N,v}(S \cup K).
\]

The proportional reduced game property states that the payoffs of every player that does not leave is the same in the reduced and original game when we apply balanced values.
Axiom 2 (Proportional reduced game property) Let \((N, v) \in G_B\) with \(n \geq 2\), and \(T \subset N\). Then \(f(T, v^f(N,v)) = f(N, v)|_T\).

Similarly to the HM-reduced game property, the proportional reduced game property can be seen as a consistency property.

Theorem 2 A solution \(F\) on \(G_B\) satisfies proportional standardness for two-player games and the proportional reduced game property if and only if it is a balanced solution.

The proof of this theorem can be found in Appendix B. Note that from every balanced solution satisfying the proportional reduced game property it follows that the reduced game of a game with balanced values also has balanced values, i.e. if \((N, v) \in G_B\) and \(x \in B(N, v)\) then \((T, v_T^x) \in G_B\) for all \(T \subset N\). We illustrate with an example.

Example 3 Consider the monotone, superadditive game \((N, v) \in G_M \cap G_A\) given by \(N = \{1, 2, 3\}\) and

\[
v(S) = \begin{cases} 
3 & \text{if } S = N \\
2 & \text{if } |S| = 2 \\
1 & \text{if } S = \{3\} \\
0 & \text{if } S \in \{\{1\}, \{2\}\},
\end{cases}
\]

with dividends

\[
\Delta_{N,v}(S) = \begin{cases} 
-2 & \text{if } S \in N \\
2 & \text{if } S = \{1, 2\} \\
1 & \text{if } S \in \{\{3\}, \{1, 3\}, \{2, 3\}\} \\
0 & \text{otherwise.}
\end{cases}
\]

The Shapley value of this game is \(\phi(N, v) = (\frac{5}{6}, \frac{5}{6}, \frac{4}{3})\). Balanced values \((x_1, x_2, x_3)\) for this game can be found by solving the system of equations

\[
x_1 = \frac{2x_1}{x_1 + x_2} + \frac{x_1}{x_1 + x_3} - \frac{2x_1}{3},
\]

\[
x_2 = \frac{2x_2}{x_1 + x_2} + \frac{x_2}{x_2 + x_3} - \frac{2x_2}{3},
\]

\[
x_3 = 1 + \frac{x_3}{x_1 + x_3} + \frac{x_3}{x_2 + x_3} - \frac{2x_3}{3}.
\]

It can be verified that \((0, \frac{3}{2}, \frac{3}{2})\), \((\frac{3}{2}, 0, \frac{3}{2})\) and \(\frac{1}{10}(15 - 3\sqrt{5}, 15 - 3\sqrt{5}, 6\sqrt{5})\) are the balanced values for \((N, v)\).

Let us focus, e.g., on the balanced value \(x = (\frac{3}{2}, 0, \frac{3}{2})\), and let \(T = \{2, 3\}\). The
proportional reduced game \((T, v^T_x)\) is given by

\[
\Delta_{T,v^T_x}(\{2\}) = \Delta_{N,v}(\{2\}) + \frac{0}{3/2} \Delta_{N,v}(\{1, 2\}) = 0 + 0 = 0,
\]

\[
\Delta_{T,v^T_x}(\{3\}) = \Delta_{N,v}(\{3\}) + \frac{3/2}{3/2 + 3/2} \Delta_{N,v}(\{1, 3\}) = 1 + \frac{1}{2} = \frac{3}{2},
\]

and

\[
\Delta_{T,v^T_x}(\{2, 3\}) = \Delta_{N,v}(\{2, 3\}) + \frac{0 + 3/2}{3/2 + 0 + 3/2} \Delta_{N,v}(\{1, 2, 3\}) = 1 - 1 = 0.
\]

So,

\[
v^T_x(S) = \begin{cases} 
0 & \text{if } S = \{2\} \\
\frac{3}{2} & \text{if } S \in \{\{3\}, \{2, 3\}\}.
\end{cases}
\]

Clearly, \((0, \frac{3}{2})\) is a balanced value for \((T, v^T_x)\).

Note that we characterized a class of single-valued solutions, containing more than one solution, while Hart and Mas-Colell (1989) characterize one specific single-valued solution. This has partly to do with the definition of the reduced game. The HM-reduced game takes account of payoffs of ‘leaving’ players in restricted games \((T, v_T)\), while in our proportional reduced game the payoffs of ‘leaving’ players in the original game \((N, v)\) matter. Because the cardinality of player sets in restricted games is lower than the cardinality of the player set in the original game, the induction hypothesis in the proof of Hart and Mas-Colell (1989) uniquely determines the payoffs of players in the HM-reduced games. This cannot be done in the proof of Theorem 2.

Hart and Mas-Colell (1989) also characterize weighted Shapley values by a ‘weighted’ standardness for two-player games property and the HM-reduced game property. However, their weighted Shapley values consider exogenously given weights which are the same for both the original and the reduced game. Besides having a class of balanced solutions, even balanced solutions need not satisfy the HM-reduced game property. This is because the endogenously determined balanced weights in the HM-reduced game can be different than the endogenous balanced weights in the original game.

5 Properties of balanced solutions

By definition the balanced solution is efficient. It even satisfies the stronger property of component efficiency. A component (see e.g. Aumann and Drèze, 1974 and Chang and
Kan, 1994) in a TU-game \((N, v) \in \mathcal{G}\) is a coalition \(B \subseteq N\) such that for all \(S \subseteq N\),
\[ v(S) = v(S \setminus B) + v(S \cap B). \]
Coalition \(B \subseteq N\) is a minimal component in \((N, v) \in \mathcal{G}\)
if \(B\) is a component in \((N, v)\) and every \(T \subset B\) is not a component in \((N, v)\). For a
detailed description see, e.g., van den Brink (1995). A solution \(F\) is component efficient if
\(x_B = v(B)\) for every \(x \in F(N, v)\) and every component \(B\) in any game \((N, v)\).

The balanced solution also satisfies the component restriction property, meaning
that for every component \(B\) in any \((N, v) \in \mathcal{G}\), if \(x, x' \in X(N, v)\) are balanced values of
\((N, v)\), then \(x'' \in X(N, v)\) given by
\[ x''_i = x_i \text{ for } i \in B \text{ and } x'_i \text{ for } i \in N \setminus B \]
is also a balanced value for \((N, v)\).

The balanced solution satisfies the null player property. Player \(i \in N\) is a null player
in \((N, v)\) if \(v(S) = v(S \setminus \{i\})\) for all \(S \subseteq N\). Solution \(F\) satisfies the null player property
if \(x_i = 0\) for all \(x \in F(N, v)\) whenever \(i\) is a null player in game \((N, v)\). For monotone,
superadditive games, the balanced solution is individually rational, i.e. \(x_i \geq v(\{i\})\) for all
\(i \in N\).

Finally, we mention that balanced values need not be core allocations nor the other
way around, where the core of game \((N, v)\) is the set of payoff vectors given by
\[ \text{Core}(N, v) = \{ x \in \mathbb{R}^n | x_N = v(N) \text{ and } x_S \geq v(S) \text{ for all } S \subseteq N \}. \]
This fact is demonstrated by the game in Example 3 which core consists of the single point
\((1, 1, 1)\). For simple games, however, every core allocation is a balanced value. (Note that
a simple game has a nonempty core if and only if there are veto players.)

**Proposition 1**  
(i) The balanced solution satisfies component efficiency, the component
restriction property and the null player property.

(ii) The balanced solution satisfies individual rationality on \(\mathcal{G}_M \cap \mathcal{G}_A\).

(iii) If \((N, v)\) is a simple game, i.e. \(v(S) \in \{0, 1\}\) for all \(S \subseteq N\), then \(\text{Core}(N, v) \subseteq
B(N, v)\).

The proof of this proposition can be found in Appendix C.
References


Appendix: Proofs

Some of the results presented in this appendix are standard (e.g., Lemma 3 and 4). We include them for the convenience of the reader.

Appendix A: Proof of Theorem 1

Throughout Appendix A we assume \((N,v)\) to be a given monotone game. We employ the following notation:

\[
X_0 = \{ x \in \mathbb{R}^n \mid x_N = v(N), \ x_i \geq 0, \ i = 1, \ldots, n \},
\]

\[
X_+ = \{ x \in \mathbb{R}^n \mid x_N = v(N), \ x_i > 0, \ i = 1, \ldots, n \},
\]

\[
\Delta(S) = \Delta_{N,v}(S) \quad \text{for} \ S \subseteq N, S \neq \emptyset,
\]

\[
h_i(x) = \sum_{S \subseteq N, i \in S, x_S \neq 0} \frac{x_i}{x_S} \Delta(S) + \sum_{S \subseteq N, i \in S, x_S = 0} \frac{1}{|S|} \Delta(S), \quad x \in X_0, \ i \in N.
\]
Further, for \( x \in X_0 \) and \( S \subseteq N \) with \( x_S > 0 \), we define
\[
k_S(x) = \sum_{T \subseteq N \atop T \supseteq S} (-1)^{|T|-|S|} x_T.
\]
Following Nowak and Radzik (1995) we establish properties of \( h_i \) and \( k_S \) needed in the sequel.

**Lemma 1**  
(i) \( h_i(x) = \sum_{S \subseteq N \atop i \in S} k_S(x)(v(S) - v(S \setminus \{i\})) \) for \( x \in X_+ \).

(ii) \( k_S(x) \geq 0 \) for every \( x \in X_+ \).

**Proof.** (i) For \( x \in X_0 \) with \( x_i > 0 \) we can write
\[
h_i(x) = \sum_{T \subseteq N \atop i \in T} x_i \Delta(T) = \sum_{T \subseteq N \atop i \in T} x_i \left( \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S) \right)
= x_i \sum_{S \subseteq N \atop i \in S} \sum_{T \subseteq N \atop T \supseteq S, i \in T} (-1)^{|T|-|S|} \frac{1}{x_T} v(S)
= x_i \sum_{S \subseteq N \atop i \in S} \sum_{T \subseteq N \atop T \supseteq S} (-1)^{|T|-|S|} \frac{1}{x_T} v(S) + x_i \sum_{S \subseteq N \atop i \notin S} \sum_{T \subseteq N \atop T \supseteq S, i \in T} (-1)^{|T|-|S|} \frac{1}{x_T} v(S)
= x_i \sum_{S \subseteq N \atop i \in S} \sum_{T \subseteq N \atop T \supseteq S} (-1)^{|T|-|S|} \frac{1}{x_T} v(S) + x_i \sum_{R \subseteq N \atop i \in R} \sum_{T \subseteq N \atop T \supseteq S} (-1)^{|T|-|R|} \frac{1}{x_T} v(R \setminus \{i\})
= x_i \sum_{S \subseteq N \atop i \in S} \sum_{T \subseteq N \atop T \supseteq S} (-1)^{|T|-|S|} \frac{1}{x_T} v(S) - x_i \sum_{S \subseteq N \atop i \in S} \sum_{T \subseteq N \atop T \supseteq S} (-1)^{|T|-|S|} \frac{1}{x_T} v(S \setminus \{i\})
= x_i \sum_{S \subseteq N \atop i \in S} k_S(x)(v(S) - v(S \setminus \{i\})).
\]

(ii) It is convenient to define the following functions \( g_i : \mathbb{R}^{i+1} \to \mathbb{R} \) by
\[
g_0(z_0) = 1, \quad z_0,
\]
and, recursively, for \( i \in \{1, \ldots, n\} \)
\[
g_i(z_0, \ldots, z_i) = g_{i-1}(z_0, \ldots, z_{i-1}) - g_{i-1}(z_0 + z_i, z_1, \ldots, z_{i-1}),
\]
and to examine their partial derivatives. Being interested only in the partial derivatives with respect to the first variable, we employ a non-standard notation: \( g_i^1 = \frac{\partial g_i}{\partial z_0} \) denotes
the derivative of \( g_i \) with respect to the first variable \( z_0 \), and, \( g_i^p = \frac{\partial^p g_i}{\partial z_0^p} \) denotes the \( p \)-th derivative of \( g_i \) with respect to the first variable \( z_0 \). These derivatives of \( g_0 \) are

\[
g_0^1(z_0) = -\frac{1}{(z_0)^2}, \quad g_0^2(z_0) = \frac{2}{(z_0)^3}, \quad g_0^3(z_0) = -\frac{6}{(z_0)^4}, \quad \ldots, \quad g_0^p(z_0) = (-1)^p \frac{p!}{(z_0)^{p+1}}.
\]

Thus, for \( z_0 > 0 \) we have that \( g_0^p(z_0) < 0 \) if \( p \) is odd, and \( g_0^p(z_0) > 0 \) if \( p \) is even. This means that \( g_0 \) is decreasing on \( \mathbb{R}_+ \), \( g_0^p(z_0) \) is increasing if \( p \) is odd, and \( g_0^p(z_0) \) is decreasing if \( p \) is even. This in turn implies that for \( z_0 > 0 \) and \( z_1 \geq 0 \) we have that,

\[
g_1(z_0, z_1) = g_0(z_0) - g_0(z_0 + z_1) \geq 0
\]

\[
g_1^p(z_0, z_1) = g_0^p(z_0) - g_0^p(z_0 + z_1) \begin{cases} \leq 0 & \text{if } p \text{ is odd,} \\ \geq 0 & \text{if } p \text{ is even.} \end{cases}
\]

In the same manner, for all \( i = 0, \ldots, n \), it can be shown that for \( z_0 > 0 \) and non-negative \( z_1, \ldots, z_n \) we have \( g_i(z_0, z_1, \ldots, z_i) \geq 0 \), \( g_i^p(z_0, z_1, \ldots, z_i) \leq 0 \) if \( p \) is odd, and \( g_i^p(z_0, z_1, \ldots, z_i) \geq 0 \) if \( p \) is even.

Having these inequalities at hand, let us elaborate on the expression of \( k_S(x) \), \( x \in X_0, x_S > 0 \). Without loss of generality, we re-number players so that \( N \setminus S = \{1, \ldots, n - |S|\} \). Then

\[
k_S(x) = \sum_{T \subseteq N, T \supseteq S} \frac{(-1)^{|T| - |S|}}{x_T} = \sum_{T \subseteq N, T \supseteq S} (-1)^{|T| - |S|} \left( \frac{1}{x_T} - \frac{1}{x_T + x_1} \right)
\]

\[
= \sum_{T \subseteq N, T \supseteq S, 1 \notin T} (-1)^{|T| - |S|} g_1(x_T, x_1) = \sum_{T \subseteq N, T \supseteq S, 1, 2 \notin T} (-1)^{|T| - |S|} (g_1(x_T, x_1) - g_1(x_T + x_2, x_1))
\]

\[
= \sum_{T \subseteq N, T \supseteq S, 1, 2 \notin T} (-1)^{|T| - |S|} g_2(x_T, x_1, x_2) = \sum_{T \subseteq N, T \supseteq S, 1, 2, 3 \notin T} (-1)^{|T| - |S|} g_3(x_T, x_1, x_2, x_3)
\]

\[
\vdots
\]

\[
= \sum_{T \subseteq N, T \supseteq S, T \cap (N \setminus S) = \emptyset} (-1)^{|T| - |S|} g_{n-|S|}(x_T, x_1, x_2, \ldots, x_{n-|S|})
\]

\[
= g_{n-|S|}(x_S, x_1, x_2, \ldots, x_{n-|S|}) \geq 0.
\]

\[
\square
\]

**Lemma 2**

(i) The mapping \( h \) is continuous on \( X_+ \).

(ii) \( h(x) \in X_0 \) for all \( x \in X_+ \).
Proof. The statement (i) is obvious. As for (ii) notice that $h$ is efficient by definition, i.e., $h(x)_N = v(N)$ for every $x \in X_0$. Using monotonicity of the game $(N,v)$ and Lemma 1 we see that $h_i(x) \geq 0$ for every $x \in X_+$ and $i \in N$. Thus we have $h(x) \in X_0$ for $x \in X_+$. □

Now we define the mapping $H : X_0 \to 2^{X_0}$ by

$$H(x) = \{ \alpha \in \mathbb{R}^n \mid \exists (x^j) \subseteq X_+ : x^j \to x, h(x^j) \to \alpha \}.$$

Lemma 3  
(i) The set $\{ (x,y) \in X_0 \times X_0 \mid y \in H(x) \}$ is closed.

(ii) $H(x)$ is a nonempty compact subset of $X_0$ for every $x \in X_0$.

(iii) $H(x) = \{ h(x) \}$ for every $x \in X_+$.

Proof.  
(i) Take sequences $(x^j), x^j \in X_+$, and $(y^j)$ such that $x^j \to x \in X_0$, $y^j \in H(x^j)$, $y^j \to y$. For each $j \in \mathbb{N}$ there exists $z^j \in X_+$ such that $||z^j - x^j|| < 1/j$ and $||h(z^j) - y^j|| < 1/j$. Then $z^j \to x$ a $h(z^j) \to y$, and consequently, $y \in H(x)$.

(ii) Fix $x \in X_0$. Since $X_0$ is compact, we have $H(x) \subseteq X_0$ by Lemma 2.(ii). Using (i) and compactness of $X_0$ we get that $H(x)$ is compact. To prove that $H(x) \neq \emptyset$ take a sequence $(x^j), x^j \in X_+$, with $x^j \to x$. By Lemma 2.(ii) the sequence $(h(x^j))$ is contained in the compact set $X_0$. Therefore there exists a convergent subsequence $(h(x^{j_k}))_{k=1}^\infty$ with a limit $\alpha \in X_0$. Thus $\alpha \in H(x)$, showing that $H(x) \neq \emptyset$.

(iii) This is a direct consequence of the fact that $h$ is continuous on $X_+$. □

Let us define now the mapping $F$ from $X_0$ to the set of all convex subsets of $X_0$ such that $F(x)$ is the convex envelope of $H(x)$ for every $x \in X_0$.

Lemma 4  
(i) $F(x)$ is a nonempty convex compact subset of $X_0$ for all $x \in X_0$.

(ii) $F(x) = \{ h(x) \}$ for all $x \in X_+$.

(iii) The set $\{ (x,y) \in X_0 \times X_0 \mid y \in F(x) \}$ is closed.

Proof.  
(i) This assertion immediately follows from Lemma 3.(ii), convexity of $X_0$ and the well known fact that the convex envelope of any compact subset of $\mathbb{R}^n$ is compact.

(ii) This part is obvious from Lemma 3.(iii).
(iii) Take sequences \((x^j), x^j \in X_0,\) and \((y^j), y^j \in X_0,\) such that \(x^j \to x \in X_0,\) \(y^j \in F(x^j),\) and \(y^j \to y.\) By Caratheodory’s theorem\(^2\) any \(y^j \in F(x^j)\) can be written as a convex combination of \(n\) elements of \(H(x).\) Since the simplex \(X_0\) is \(n - 1\) dimensional, there are \(\alpha^j_1, \ldots, \alpha^j_n \in [0, 1]\) and \(y^{j,1}, \ldots, y^{j,n} \in H(x^j)\) such that
\[
\alpha^j_1 y^{j,1} + \cdots + \alpha^j_n y^{j,n} = y^j \quad \text{and} \quad \sum_{s=1}^{n} \alpha^j_s = 1.
\]
Going to subsequences, if necessary, we may assume that \(\alpha^j_s \to \alpha_s \in [0, 1],\) and \(y^{j,s} \to y^{\infty,s}.\) Then
\[
\alpha_1 y^{\infty,1} + \cdots + \alpha_n y^{\infty,n} = y \quad \text{and} \quad \sum_{s=1}^{n} \alpha_s = 1.
\]
Since the graph of \(H\) is closed by Lemma 3.(i), we have that \(y^{\infty,s} \in H(x),\) and thus \(y \in F(x).\) \(\square\)

Since we showed that \(F\) and its domain satisfy the assumptions of Kakutani’s theorem\(^3\) (see Kakutani (1941) or e.g., Franklin (1980)) we have the following corollary.

**Corollary 1** There exists a \(x^* \in X_0\) such that \(x^* \in F(x^*).\)

However, we are interested in fixed points of the mapping that assigns to every \(x \in X_0\) the set of fixed points of function \(h.\) For that we need continuity of \(h\) at \(x^*\).

**Lemma 5** Let \(x^* \in X_0\) be such that \(x^* \in F(x^*).\) Then \(h: X(N, v) \to X(N, v)\) is continuous at \(x^*\).

**Proof.** We distinguish two cases.

a) If \(x^* \in X_+\) then \(x^* \in F(x^*) = \{h(x^*)\},\) and consequently, \(x^* = h(x^*).\)

b) Suppose now that \(x^* \in X_0 \setminus X_+.\) Then there are elements \(z^1, \ldots, z^p \in H(x^*)\) and \(\beta_1, \ldots, \beta_p \in (0, 1]\) such that \(\sum_{s=1}^{p} \beta_s = 1\) and
\[
\beta_1 z^1 + \cdots + \beta_p z^p = x^*.
\]
\(\text{(14)}\)

Denote \(Q = \{i \in N \mid x_i^* = 0\}.\) Since \(z^j_i \geq 0, j = 1, \ldots, p,\) the Equation (14) guarantees that \(z^j_i = 0\) for every \(i \in Q, j \in \{1, \ldots, p\}.\) Let us simplify the notation by setting

---

\(^2\)Caratheodory’s theorem asserts that each element of the convex envelope of a set \(M \subseteq \mathbb{R}^{n-1}\) can be written as a convex combination of \(n\) elements of the set \(M.\)

\(^3\)This theorem says that any upper hemicontinuous correspondence \(F\) from a non-empty, compact convex subset \(S\) of \(\mathbb{R}^n\) to itself such that \(F(x)\) is convex, closed and non-empty for all \(x \in S,\) has a fixed point, i.e. then there exists an \(x^* \in S\) such that \(x^* \in F(x).\)
\[ z := z^1. \] Since \( z \in H(x^*) \) there exists a subsequence \( (x^j) \), \( x^j \in X_+ \), such that \( x^j \to x^* \) and \( h(x^j) \to z \).

For \( i \in Q \) we have \( \lim_{j \to \infty} x^j_i = 0 \) and \( \lim_{j \to \infty} h_i(x^j) = 0 \). Further we get

\[
\sum_{i \in Q} h_i(x^j) = \sum_{i \in Q} \left( \sum_{S \subseteq N} \frac{x^j_i}{x_S} \Delta(S) \right) = \sum_{i \in Q} \left( \sum_{S \subseteq N} \frac{x^j_i}{x_S} \Delta(S) \right) + \sum_{i \in Q} \left( \sum_{S \subseteq Q} \frac{x^j_i}{x_S} \Delta(S) \right) = \sum_{i \in Q} \left( \sum_{S \subseteq N} \frac{x^j_i}{x_S} \Delta(S) \right) + \sum_{S \subseteq Q} \Delta(S) = \sum_{i \in Q} \left( \sum_{S \subseteq N} \frac{x^j_i}{x_S} \Delta(S) \right) + v(Q).
\]

The limit of the left side is

\[
\lim_{j \to \infty} \sum_{i \in Q} h_i(x^j) = 0,
\]

and since \( S \setminus Q \neq \emptyset \) implies \( x^*_S > 0 \) also

\[
\lim_{j \to \infty} \sum_{i \in Q} \left( \sum_{S \subseteq N} \frac{x^j_i}{x_S} \Delta(S) \right) = 0.
\]

Therefore, by monotonicity of \((N, v)\) we have that \( v(Q) = 0 \) and so \( v(S) = 0 \) for all \( S \subseteq Q \).

Consequently, \( \Delta(S) = 0 \) for all \( S \subseteq Q \). Then

\[
h_i(x) = \sum_{S \subseteq N, i \in S, x_S \neq 0} \frac{x_i}{x_S} \Delta(S) + \sum_{S \subseteq N, i \in S, x_S = 0} \frac{1}{|S|} \Delta(S)
\]

\[
= \sum_{S \subseteq N, i \in S, S \setminus Q \neq \emptyset, x_S \neq 0} \frac{x_i}{x_S} \Delta(S) + \sum_{S \subseteq N, i \in S, S \setminus Q \neq \emptyset, x_S = 0} \frac{1}{|S|} \Delta(S), \quad x \in X_0.
\]
Since $x_i^* > 0$ for all $i \in N \setminus Q$, there exists a neighborhood $V$ of $x^*$ such that for every $x \in X_0 \cap V$ and $S \subseteq N$ with $S \setminus Q \neq \emptyset$ we have $x_S > 0$ and

$$h_i(x) = \sum_{S \subseteq N; i \in S; x_S \neq 0} \frac{x_i}{x_S} \Delta(S).$$

From this we conclude that $h$ is continuous at $x^*$. □

From continuity of $h$ at fixed point $x^*$ of $F$ it follows that $x^* \in F(x^*) = \{h(x^*)\}$, and thus $x^* = h(x^*)$, showing that $B(N, v) \neq \emptyset$.

Appendix B: Proof of Theorem 2

Every balanced solution satisfying proportional standardness for two-player games is easy to verify from Example 1.

To show that every balanced solution satisfies the proportional reduced game property, suppose that $(N, v) \in \mathcal{G}_B, x = f(N, v)$ and $T \subset N$. We have to show that $x|_T$ is a balanced value for the reduced game $(T, v^T_x)$, i.e.

$$\sum_{S \subseteq T; i \in S; x_S \neq 0} \frac{x_i}{x_S} \Delta_{T, v^T_x}(S) + \sum_{S \subseteq T; i \in S; x_S = 0} \frac{1}{|S|} \Delta_{T, v^T_x}(S) = x_i \text{ for all } i \in T. \tag{15}$$

By definition of $\Delta_{T, v^T_x}$, the first term of this expression is

$$\sum_{S \subseteq T; i \in S; x_S \neq 0} \frac{x_i}{x_S} \Delta_{T, v^T_x}(S) = \sum_{S \subseteq T; i \in S; x_S \neq 0} \frac{x_i}{x_S} \Delta_{N, v}(S) +$$

$$+ \sum_{S \subseteq T; i \in S; x_S \neq 0} \sum_{K \subseteq T^c; K \neq \emptyset; x_{S \cup K} \neq 0} \frac{x_i}{x_S} \cdot x_S \cdot \Delta_{N, v}(S \cup K) +$$

$$+ \sum_{S \subseteq T; i \in S; x_S \neq 0} \sum_{K \subseteq T^c; K \neq \emptyset; x_{S \cup K} = 0} \frac{x_i}{x_S} \cdot \frac{|S|}{|S \cup K|} \Delta_{N, v}(S \cup K).$$

For $x \in X_0(N, v)$ and $S \subseteq N$, $x_S = 0$ implies that $x_i = 0$ for all $i \in S$. It follows that $x_{S \cup K} = 0$ implies $x_S = 0$, and thus the last term in the expression above disappears, so the first term in (15) can be written as

$$\sum_{S \subseteq T; i \in S; x_S \neq 0} \frac{x_i}{x_S} \Delta_{T, v^T_x}(S) = \sum_{S \subseteq T; i \in S; x_S \neq 0} \frac{x_i}{x_S} \Delta_{N, v}(S) +$$

$$+ \sum_{S \subseteq T; i \in S; x_S \neq 0} \sum_{K \subseteq T^c; K \neq \emptyset; x_{S \cup K} \neq 0} \frac{x_i}{x_S} \cdot \Delta_{N, v}(S \cup K). \tag{16}$$
Similarly, the second term of expression (15) can be written as

\[
\sum_{S \subseteq T, i \in S} \frac{1}{|S|} \Delta_{T, v_i}^f(S) = \sum_{S \subseteq T, i \in S} \frac{1}{|S|} \Delta_{N,v}^f(S) + \sum_{S \subseteq T, i \in S} \frac{1}{|S|} \sum_{K \subseteq T^c, K \neq \emptyset} \frac{1}{|S \cup K|} \Delta_{N,v}(S \cup K) + \sum_{S \subseteq T, i \in S} \frac{1}{|S|} \sum_{K \subseteq T^c, K \neq \emptyset} \frac{1}{|S \cup K|} \Delta_{N,v}(S \cup K).
\]

Summing up (16) and (17) (and using the facts that \(x_S = 0 \Rightarrow x_{S \cap T} = 0\), and \(x_{S \cap T} \neq 0 \Rightarrow x_S \neq 0\)) gives

\[
\sum_{S \subseteq T, i \in S} \frac{x_i}{|S|} \Delta_{T, v_i}^f(S) + \sum_{S \subseteq T, i \in S} \frac{1}{|S|} \Delta_{T, v_i}^f(S) = \sum_{S \subseteq T, i \in S} \frac{x_i}{|S|} \Delta_{N,v}(S) + \sum_{S \subseteq T, i \in S} \frac{1}{|S|} \Delta_{N,v}(S) + \sum_{S \subseteq N, S \cap T = \emptyset, x_S = 0} \frac{1}{|S|} \Delta_{N,v}(S) = \sum_{S \subseteq N, i \in S} \frac{x_i}{|S|} \Delta_{N,v}(S) = x_i.
\]

So (15) holds, showing that balanced solutions satisfy the proportional reduced game property.

It remains to show that \(f\) satisfying proportional standardness for two-player games and the proportional reduced game property, implies that \(f\) is a balanced solution. Therefore, suppose that \(f\) satisfies these properties. The proof proceeds by induction on the cardinality of \(N\).

If \(n = 1\), i.e. \(N = \{i\}\), then, similarly as done in Hart and Mas-Colell (1989), we extend the game \((N, v)\) by a null player \(j\). Thus we obtain a two-player game \(\{i, j\}, w\) given by \(w(\{i\}) = w(\{i, j\}) = v(\{i\})\) and \(w(\{j\}) = 0\). The proportional reduced game \((\{i\}, w_{\{i\}}^f)\) is then determined by the dividend \(\Delta_{\{i\}, w_{\{i\}}^f}(\{i\}) = \Delta_{N,v}(\{i\}) = v(\{i\})\), which means that \(w_{\{i\}}(\{i\}) = v(\{i\})\). The proportional reduced game \((\{i\}, w_{\{i\}}^f)\) is thus identical with the game \((N, v)\). Since \((\{i, j\}, w) \in \mathcal{G}_B\) (see Example 1), proportional standardness for two-player games implies that \(f(\{i, j\}, w)\) consists of a unique payoff vector \((v(\{i\}), 0)\). From the proportional reduced game property it then follows that \(f(N, v) = f(\{i\}, w_{\{i\}}^f) = (v(\{i\}), 0)\).
If \( n = 2 \), then proportional standardness ensures that \( f(N, v) \) is a balanced value.

Proceeding by induction, assume that \( f(N, v) \) is a balanced value for any \((N', v') \in \mathcal{G}_B\) with \(|N'| < n\). We should prove that \( f(N, v) \) is a balanced value of \((N, v)\). To show this, consider \( T = \{i, j\} \), where \( i \in N \) and \( j \in N \setminus \{i\} \) are two arbitrary players from \( N \). The proportional reduced game property implies that if \( x = f(N, v) \) then \( x|_T \) is a balanced value of \((T, v^T_0)\). In a similar way as before in proving that a balanced solution satisfies the proportional reduced game property, it can be shown that

\[
x_i = \sum_{S \subseteq T, i \in S} \frac{x_i}{x_S} \Delta_{T, v^T_0}(S) + \sum_{S \subseteq T, i \in S} \frac{1}{|S|} \Delta_{T, v^T_0}(S) = \\
= \sum_{S \subseteq N, i \in S} \frac{x_i}{x_S} \Delta_{N, v}(S) + \sum_{S \subseteq N, i \in S} \frac{1}{|S|} \Delta_{N, v}(S).
\]

Appendix C: Proof of Proposition 3

(i) The proof of this part is straightforward, and therefore is left to the reader. (Note that the properties are obviously satisfied for games with no balanced values.)

(ii) Let \((N, v)\) be a monotone, superadditive game. By Theorem 1, \( B(N, v) \neq \emptyset \). We use the same notation as in the previous appendices. Using Lemma 1 and superadditivity of \((N, v)\) we get for \( x \in X_0 \) with \( x_i > 0 \) the following estimates.

\[
h_i(x) = x_i \sum_{S \subseteq N, i \in S} k_S(x) (v(S) - v(S \setminus \{i\})) \\
\geq x_i \sum_{S \subseteq N} (k_S(x) \cdot v(\{i\})) = x_i \cdot v(\{i\}) \cdot \sum_{S \subseteq N} k_S(x).
\]

The term \( \sum_{S \subseteq N} k_S(x) \) can be rewritten as follows

\[
\sum_{S \subseteq N} k_S(x) = \sum_{T \subseteq N} \sum_{S \subseteq T, i \in S} (-1)^{|T| - |S|} \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \equiv 22
Further we compute
\[
\sum_{S \subseteq T, i \in S} (-1)^{|S|} = \sum_{R \subseteq T \setminus \{i\}} (-1)^{|R|-1} = - \sum_{n=0}^{|T|-1} \sum_{R \subseteq T \setminus \{i\}} (-1)^{-n} \\
= - \left(\binom{|T| - 1}{0} - \binom{|T| - 1}{1} + \cdots + (-1)^{|T|-1} \binom{|T| - 1}{|T| - 1}\right) \\
= \begin{cases} 
-1 & \text{for } |T| = 1, \\
-(1 - 1)^{|T|-1} = 0 & \text{for } |T| > 1.
\end{cases}
\]
Thus we get
\[
\sum_{S \subseteq N} k_S(x) = \sum_{T \subseteq N} x_T \frac{(-1)^{|T|+1}}{x_i} = \frac{1}{x_i}, \quad (19)
\]
The inequality (18) and the identity (19) give \( h_i(x) \geq v(\{i\}) \) for \( x \in X_0 \) with \( x_i > 0 \).

Now let \( x^* \) be a balanced value of the game \((N, v)\), i.e., \( h(x^*) = x^* \). Denote \( Q = \{ i \in N \mid x^*_i = 0 \} \). If \( i \in N \setminus Q \), then we have \( x^*_i > 0 \) and so \( x^*_i = h_i(x^*) \geq v(\{i\}) \).

Further we have
\[
0 = \sum_{i \in Q} x^*_i = \sum_{i \in Q} h_i(x^*) = \sum_{i \in Q} \left( \sum_{S \subseteq N, i \in S, x_S \neq 0} x^*_i x_S \Delta(S) + \sum_{S \subseteq N, i \in S, x_S = 0} \frac{1}{|S|} \Delta(S) \right) \\
= \sum_{i \in Q} \left( \sum_{S \subseteq N, i \in S, S \setminus \{\{i\}\} \neq 0} x^*_i x_S \Delta(S) + \sum_{S \subseteq Q, i \in S} \frac{1}{|S|} \Delta(S) \right) = \sum_{i \in Q} \sum_{S \subseteq Q} \frac{1}{|S|} \Delta(S) = \sum_{S \subseteq Q} \Delta(S) = v(Q).
\]
By monotonicity and superadditivity this gives \( v(\{i\}) = 0 \) for all \( i \in Q \), and we have \( x^*_i = v(\{i\}) \) for every \( i \in Q \), showing individual rationality of the balanced solution for superadditive games.

(iii) Finally, if \((N, v)\) is a simple game then \( \text{Core}(N, v) \neq \emptyset \) if and only if \((N, v)\) contains veto players, i.e. there is a coalition \( T \subseteq N \) such that \( v(S) = 0 \) for all \( S \supseteq T \). If \( \text{Core}(N, v) = \emptyset \) then obviously \( \text{Core}(N, v) \subseteq B(N, v) \). Otherwise, \( \text{Core}(N, v) = \{ x \in \mathbb{R}^N_+ \mid x_T = 1 \text{ and } x_i = 0 \text{ for all } i \in N \setminus T \} \), where \( T \) is the coalition of veto players. Take \( x \in \text{Core}(N, v) \). Since \( x_S = 0 \) implies \( S \not\supseteq T \) and thus \( \Delta_{N,v}(S) = 0 \), we have
\[
h_i(x) = \sum_{S \subseteq N, i \in S, x_S \neq 0} x_i x_S \Delta_{N,v}(S) + \sum_{S \subseteq N, i \in S, x_S = 0} \frac{1}{|S|} \Delta_{N,v}(S) = \sum_{S \subseteq N, i \in S, x_S \neq 0} x_i x_S \Delta_{N,v}(S).
\]
If $i \in N \setminus T$, then $x_i = 0$ and thus $h_i(x) = 0$. If $i \in T$, then

$$h_i(x) = \sum_{S \subseteq N, i \in S} \frac{x_i}{x_S} \Delta_{N,v}(S) = \sum_{S \subseteq N, S \supseteq T} \frac{x_i}{x_S} \Delta_{N,v}(S) = x_i \sum_{S \subseteq N} \Delta_{N,v}(S) = x_i.$$ 

In any case $h_i(x) = x_i$, and thus $x \in F(N,v)$. 

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