Ordering infinite utility streams: completeness at the cost of a non-Ramsey set

Luc Lauwers

Center for Economic Studies, K.U.Leuven

Naamsestraat 69, B-3000 Leuven, BELGIUM

Luc.Lauwers@econ.kuleuven.be *

This version: February 15, 2007

Abstract. The existence of a complete, Paretian, and finite anonymous ordering in the set of infinite utility streams implies the existence of a non-Ramsey set. In set theory, the existence of a non-Ramsey set involves the axiom of choice. Therefore, each Paretian and finite anonymous social welfare relation either is incomplete or does not have an explicit description. Furthermore, the strongest anonymity condition that is compatible with Pareto involves a free ultrafilter on the lattice of partitions of positive integers. These results confirm a conjecture of Fleurbaey and Michel (2003) and further disentangle the impossibility theorems of Diamond (1965) and of Basu and Mitra (2003).

KEYWORDS: social welfare orderings, completeness, anonymity, equity, Pareto, ultrafilter, partitions, Ramsey set, axiom of choice.

1 Introduction

Does there exist a relation in the set of infinite utility streams that combines transitivity, completeness, Pareto, and finite anonymity? This problem is at the heart of the axiomatic approach to social choice when the population is infinite, and finds its origin in Frank

*I am deeply indebted to Marc Fleurbaey who guided me into this problem already in 1996. I thank Marc for the many discussions and for the constant support. I also thank Norbert Brunner, Johan Quaegebeur, and Luc Van Liedekerke for helpful conversations. This paper was presented in the workshop "Intergenerational equity in climate negotiations, overlapping generations and social welfare" organized by Claude d'Aspremont and Thierry Bréchet (CORE, April 27-28, 2006). This text presents research results of the Belgian Programme of Interuniversity Poles of Attraction initiated under the Science Policy Programmes of the Prime Minister's Office, Belgium. The scientific responsibility is assumed by its authors.

Plumpton Ramsey's (1928a) firm rejection to discount later enjoyments in comparison with earlier ones.[1] Following Koopmans (1960), Diamond (1965) establishes the celebrated result that there does not exist a social welfare relation (SWR) that combines completeness, Pareto, finite anonymity, and continuity. This impossibility result has been sharpened by, among others, Fleurbaey and Michel (2003), and Basu and Mitra (2003). Basu and Mitra show that Diamond's result persists even when continuity is dropped and representability is imposed: a Paretian social welfare function does not satisfy finite anonymity.

On the other hand, Svennson (1980) established the general possibility result that one can find a complete, Paretian, and finite anonymous SWR. His *construction* however involves Szpilrajn's lemma. Although Szpilrajn's lemma is weaker than the axiom of choice (AC), it has a nonconstructive component and removes a lot of the attractiveness of Svennson's criterion.[2]

Fleurbaey and Michel (2003)[3] summarize both tracks—incompatibilities versus possibilities through Szpilrajn—as follows:

> "In view of our results, we propose the conjecture there is no *explicit* (that is, avoiding the axiom of choice or similar contrivances) description of an ordering which satisfy weak Pareto and indifference to finite permutations. ...
>
> If this conjecture is true, it means that for all practical purposes, weak Pareto and indifference to finite permutations are incompatible. Indeed, take any model of optimal growth. An explicit description of the ordering describing social preferences would generally be needed in order to compute the optimal path. The fact that, based on the axiom of choice or free ultrafilters, we know that orderings satisfying weak Pareto and indifference to finite permutations exist is of no use in this practical context."

This note investigates their conjecture. We obtain two results.

First, we consider the reflexive and transitive closure of the relation generated by the axioms of Pareto and finite anonymity. We expand this partial ordering by strengthening the anonymity condition. Apparently, the strongest anonymity condition compatible with Pareto allows for a complete SWR but involves the existence of an ultrafilter on the lattice of partitions of the set of positive integers. Recall that anonymity conditions are expressed through groups of permutations. Since only cyclic permutations are compatible with Pareto (Mitra and Basu, 2006), this first result is obtained through the study of maximal (for inclusion) groups within the set of cyclic permutations.

The nonconstructiveness of ultrafilters is well known. Jehne and Klinge (1977, p209), for example, state that free ultrafilters on the set of positive integers are so highly uncon-

[1] This view, formally expressed by an anonymity condition, was already defended by Sidgwick (1907).

[2] Let ZFC denote the axiomatic set theory of Zermelo-Fraenkel augmented with AC (Jech, 1978, p1; Fraenkel et al, 1973). We say that an object is "nonconstructive" or "does not have an explicit description" in case its existence needs the use of AC.

[3] A working paper version was presented in Montreal (May 1996) and in Maastricht (July 1996) and appeared as "Document de travail THEMA 9711".

structive that they cannot be distinguished from one another. As each ultrafilter on the lattice of partitions is free, their statement also holds for the maximal groups of cyclic permutations.

Second, we show that each complete, Paretian, and finite anonymous SWR generates a non-Ramsey set. The concept, named after Frank Plumpton Ramsey, is as follows. Let I be an infinite set, let n be a natural number, and let $[I]^n$ be the collection of all subsets of I with exactly n elements. Ramsey (1928b) proved that if the collection $[I]^n$ is partitioned into a finite number of pieces, then there exists an infinite subset J of I such that $[J]^n$ is included in one of those pieces.[4] Ramsey's theorem does not hold for n equal to infinity. There exists a subset R of $[I]^\infty$ such that for each infinite subset J of I, the collection $[J]^\infty$ intersects both R and its complement $[I]^\infty - R$ (the partition $\{R\,,\,[I]^\infty - R\}$ violates the Ramsey property). Such a set R is said to be non-Ramsey. Mathias (1977) showed that the existence of non-Ramsey sets involves AC. This establishes our second result: a Paretian and finite anonymous SWR is either incomplete or has no explicit description.

The next section collects preliminaries. Section 3 shows how to use ultrafilters on the set of positive integers in order to obtain SWRs that combine finite anonymity, Pareto, and completeness. We discuss the relationships between AC, the ultrafilter theorem, and Szpilrajn's lemma. Section 4 introduces the concept of ultrafilters on the lattice of partitions. Section 5 prepares the first result and shifts the focus to ranking subsets rather than utility streams. Section 6 develops the first result: a maximal anonymity condition involves an ultrafilter on the lattice of paritions (Theorem 1). Section 7 concentrates on a particular set of permutations: the group of fixed step permutations in the class of variable step permutations. Section 8 reconsiders Svennson possibility result and recommends to use ultrafilters rather than AC in order to extend a partial ordering. Section 9 recalls the definition of a Ramsey set and proves the second result (Theorem 2).

2 Preliminaries

We recall the notions of a social welfare relation and of cyclic permutations. Mitra and Basu (2006) established a link between the Pareto axiom and the set of cyclic permutations. We provide a different proof of their result.

2.1 Social welfare relations

Let $\mathbb{N}_0 = \{1, 2, 3, \ldots\}$ denote the set of positive integers, $\mathbb{R}$ the set of real numbers, and $\mathbb{Q}$ the set of rational numbers. Let $Y \subseteq \mathbb{R}$ be the set of all possible utility levels. We follow Basu and Mitra (2003) and assume that Y has at least two distinct elements, say, 0 and 1. The set $X = Y^{\mathbb{N}_0}$ collects all possible utility streams and is called the domain. An infinite utility stream x is a vector in X. Each x in X can be viewed as a map from $\mathbb{N}_0$ to

[4] This theorem initiated the so-called 'Ramsey theory', an area within combinatorial set theory.

Y, associating with each t in $\mathbb{N}_0$ the element x_t in Y. Each utility stream x in $\{0,1\}^{\mathbb{N}_0}$ is identified with the subset $\{\, t \,|\, x_t = 1 \,\}$ of $\mathbb{N}_0$. Let $\mathcal{S}$ collect all subsets of $\mathbb{N}_0$. Due to the identification of subsets of $\mathbb{N}_0$ with their indicator functions, we abuse language and say that $\mathcal{S}$ is a subset of X. Vector inequalities are denoted $\leq$, $<$, and $\ll$. Set inclusions are denoted $\subseteq$ and $\subset$.

A social welfare relation (SWR) is a binary relation, $\precsim$, in the domain X, which is reflexive and transitive. The symmetric and the asymmetric component of the SWR $\precsim$ are denoted by $\sim$ and $\prec$. The SWR $\precsim_1$ is a subrelation to a SWR $\precsim_2$ if for each x and y in X we have (i) $x \precsim_1 y$ implies $x \precsim_2 y$ and (ii) $x \prec_1 y$ implies $x \prec_2 y$.

A permutation π is a one-to-one map from $\mathbb{N}_0$ to $\mathbb{N}_0$. For each x in X, the composite map $x \circ \pi$ is a map from $\mathbb{N}_0$ to Y and can be written as the infinite utility stream

$$x \circ \pi \;=\; \big(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(t)}, \ldots\big)\,.$$

Let $\mathrm{Sym}(\mathbb{N}_0)$ collect all permutations on $\mathbb{N}_0$. The set $\mathrm{Sym}(\mathbb{N}_0)$ when equipped with the composition operation becomes a group. The next definition collects three Pareto axioms and two concepts related to permutations.

Definition 1.

- A SWR $\precsim$ satisfies the Pareto axiom if for each x and y in X we have that $x < y$ implies $x \prec y$.
- A SWR $\precsim$ satisfies the weak Pareto axiom if for each x and y in X,
  - we have that $x \leq y$ implies $x \precsim y$,
  - and that $x \ll y$ implies $x \prec y$.
- A SWR $\precsim$ satisfies the intermediate Pareto axiom if for each x and y in X,
  - we have that $x \leq y$ implies $x \precsim y$,
  - and that $x < y$ and $x_i < y_i$ for infinitely many i in $\mathbb{N}_0$ implies $x \prec y$.
- Let $\mathcal{Q}$ be a class of permutations. A SWR $\precsim$ satisfies $\mathcal{Q}$-anonymity if for each π in $\mathcal{Q}$ and for each x in X we have $x \sim x \circ \pi$.
- Let $\precsim$ be a SWR. The set of permissible permutations is defined to be

$$\Pi(\precsim) \;=\; \Big\{\, \pi \text{ in } \mathrm{Sym}(\mathbb{N}_0) \;\Big|\; \text{for each } x \text{ in } X \text{ we have } x \circ \pi \sim x \,\Big\}.$$

The Pareto axiom postulates sensitivity in each coordinate. The intermediate Pareto axiom postulates sensitivity in each infinite set of coordinates. This intermediate version is useful when ranking subsets of $\mathbb{N}_0$, where the imposition of weak Pareto only demands that the full set $\mathbb{N}_0$ is strictly larger than the empty set $\varnothing$.

With respect to anonymity, we only consider classes of permutations that include the group of finite permutations. Hereby, the permutation π is said to be finite if there exists a T in $\mathbb{N}_0$ such that $\pi(t) = t$ for each $t \geq T$. Let $\mathcal{Q}_{\text{fin}}$ collect all finite permutations. A SWR is said to be finite anonymous if it satisfies $\mathcal{Q}_{\text{fin}}$-anonymity.

2.2 Cyclic permutations

Let π be a permutation on the set $\mathbb{N}_0$. The vector $(k, \pi(k), \pi^2(k), \pi^3(k), \ldots)$ is said to be the cycle generated by π on k. Each permutation can be written as a succession of cycles on disjoint sets (Hall, 1976, Chapter 5). For example the permutation

$$\pi_1 = (1,2)(3,4)(5,6)\cdots(2n-1,2n)\cdots$$

switches the odd and even numbers, for each n in $\mathbb{N}_0$ the number $2n-1$ is mapped upon $2n$ and $2n$ is mapped upon $2n-1$. The final element in a cycle is mapped upon the first element in that cycle. The permutation

$$\pi_2 = (1)(2,3)(4,5)\cdots(2n,2n+1)\cdots$$

keeps the number 1 fixed and then switches the even and odd numbers. A permutation on $\mathbb{N}_0$ might generate a cycle of infinite length. The permutation

$$\pi_3 = (\ldots, 9, 7, 5, 3, 1, 2, 4, 6, 8, \ldots).$$

maps 1 upon 2. Furthermore, π_3 maps an even number upon its even successor and an odd number upon its odd predecessor, as such $\pi_3(123) = 121$ and $\pi_3(100) = 102$. We keep the references π_1, π_2, and π_3 throughout this note.

The decomposition of a permutation into pairwise disjoint cycles is unique, except for the order in which the cycles are written, also within each cycle the numbers are allowed to be permuted cyclically. For example, the permutations $(1,2)(3)(4,5,6,7)$ and $(3)(1,2)(5,6,7,4)$ coincide.

A permutation representable by an infinite sequence of finite cycles is said to be cyclic. Alternatively (Mitra and Basu, 2006), the permutation π is cyclic if for each n in $\mathbb{N}_0$ there exists a k in $\mathbb{N}_0$ such that

$$\pi^k(n) \;=\; \underbrace{\pi \circ \pi \circ \cdots \circ \pi}_{k \text{ times}}(n) \;=\; n.$$

The period k might be different for different values of n. Hence, a cyclic permutation π is non-wandering in the sense that for each n in $\mathbb{N}_0$ the sequence $\pi(n)$, $\pi^2(2)$, $\pi^3(n)$, $\ldots$ returns to n after a finite numbers of iterations.

Each permutation partitions the set $\mathbb{N}_0$: present the permutation as a juxta position of cycles and replace the brackets (and) by { and }. Each cyclic permutation partitions the set $\mathbb{N}_0$ into an infinite sequence of finite sets. For example, the partition induced by the permutation π_1 is equal to

$$\text{Part}(\pi_1) = \Big\{ \{1,2\}, \{3,4\}, \ldots, \{2n-1, 2n\}, \ldots \Big\}.$$

The set of all cyclic permutations is denoted by $\mathcal{P}$. Obviously, finite permutations are cyclic. The next lemma highlights the main motivation to study cyclic permutations. The lemma already appeared in Mitra and Basu (2006, Lemma 1). We provide a different proof.

Lemma 1. A permutation π is cyclic if and only if there is no x in X satisfying $x < x \circ \pi$.

Proof. The only-if-part is straightforward. If the permutation π is cyclic, then it can be decomposed as an infinite juxta position of permutations on finite sets. Each permutation on a finite set is unable to conflict with the Pareto principle.

The if-part (if there is no conflict with Pareto, then the permutation is cyclic) is done by contraposition. Hence, consider a permutation π with an infinite cycle at m in $\mathbb{N}_0$:

$$(\ldots, \pi^{-4}(m), \pi^{-3}(m), \pi^{-2}(m), \pi^{-1}(m), m, \pi^1(m), \pi^2(m), \pi^3(m), \pi^4(m), \ldots).$$

Relabel this cycle (let 1 denote m) to obtain the cycle π_3 and consider the following table:

$$\begin{array}{rl} \pi_3 = & (\ldots, \; 9, \; 7, \; 5, \; 3, \; 1, \; 2, \; 4, \; 6, \; 8, \; \ldots), \\ x = & (\ldots, \; 0, \; 0, \; 0, \; 0, \; 0, \; 1, \; 1, \; 1, \; 1, \; \ldots), \\ y = x \circ \pi_3 = & (\ldots, \; 0, \; 0, \; 0, \; 0, \; 1, \; 1, \; 1, \; 1, \; 1, \; \ldots). \end{array}$$

The first line in this table is a cycle of infinite length. The second line presents an infinitely long utility stream in X. This utility stream is made up of two sequences, a sequence of 'ones' is attached to the even positions ($x_{2n} = 1$) and a sequence of zeros is attached to the odd positions ($x_{2n-1} = 0$). The final line presents the permuted utility stream $y = x \circ \pi_3$ (recall that $y_i = x_{\pi(i)}$). The utility stream y dominates x (indeed, $x_1 < y_1$). $\square$

The above proof only uses utility streams made up with zeros and ones (such a utility stream is an indicator function of a subset of $\mathbb{N}_0$). Hence, we obtain:

Corollary 1. A permutation π on $\mathbb{N}_0$ is cyclic if and only if there is no set S in $\mathcal{S}$ satisfying the strict inclusion $S \subset \pi(S)$.

In case the domain $X = Y^{\mathbb{N}_0}$ is sufficiently rich ($\mathbb{Q} \cap [0,2] \subseteq Y$), the infinite cycle π_3 generates a stronger domination result. There exists a stream z such that $z \ll (z \circ \pi_3)$:

$$\begin{array}{rl} \pi_3 = & (\ldots, \; 9, \; 7, \; 5, \; 3, \; 1, \; 2, \; 4, \; 6, \; 8, \; \ldots), \\ z = & (\ldots, \; \tfrac{1}{9}, \; \tfrac{1}{7}, \; \tfrac{1}{5}, \; \tfrac{1}{3}, \; 1, \; 2\text{-}\tfrac{1}{2}, \; 2\text{-}\tfrac{1}{4}, \; 2\text{-}\tfrac{1}{6}, \; 2\text{-}\tfrac{1}{8}, \; \ldots), \\ z \circ \pi_3 = & (\ldots, \; \tfrac{1}{7}, \; \tfrac{1}{5}, \; \tfrac{1}{3}, \; 1, \; 2\text{-}\tfrac{1}{2}, \; 2\text{-}\tfrac{1}{4}, \; 2\text{-}\tfrac{1}{6}, \; 2\text{-}\tfrac{1}{8}, \; 2\text{-}\tfrac{1}{10}, \; \ldots). \end{array}$$

Lemma 1, thus, holds when Pareto is weakened to intermediate Pareto or weak Pareto.

Corollary 2. Let Y include $\mathbb{Q} \cap [0, 2]$. A permutation π is cyclic if and only if there is no utility stream x in $X = Y^{\mathbb{N}_0}$ satisfying $x \ll x \circ \pi$.

Lemma 1 and Corollary 2 indicate that each infinite cycle conflicts with the Pareto axioms. We summarize. Let Y be sufficiently rich. Let $\mathcal{Q}$ be a class of permutations. Then

$\mathcal{Q}$-anonymity and Pareto are compatible	$\iff$	$\mathcal{Q}$-anonymity and weak Pareto are compatible	$\iff$	the class $\mathcal{Q}$ only contains cyclic permutations.

Apparently, there is no trade-off between the three Pareto axioms and anonymity. As soon Q-anonymity and weak Pareto are compatible, the Q-anonymity axiom does not conflict with stronger Pareto conditions.

We close this section with a result of Mitra and Basu (2006, Proposition 3). For a group $\mathcal{Q}$ of cyclic permutations, we define the relation $\precsim_{\mathcal{Q}}$ as follows. For each x and y in X, we have

$$x \precsim_{\mathcal{Q}} y \qquad \text{if and only if} \qquad \text{there is a } \pi \text{ in } \mathcal{Q} \text{ such that } x \circ \pi \leq y.$$

Proposition 1 (Mitra and Basu, 2006)**.** Let $\mathcal{Q}$ be a group of cyclic permutations. Then, the relation $\precsim_{\mathcal{Q}}$ in X is reflexive, transitive, and Paretian. Furthermore, $\Pi(\precsim_{\mathcal{Q}}) = \mathcal{Q}$.

3 Ultrafilters and completeness

We provide the definitions of a filter and an ultrafilter, discuss how AC is involved in the existence of a free ultrafilter, indicate how a free ultrafilter can be used to prove Szpilrajn's lemma, and provide some examples of complete SWRs on the domain X.

Let S be a set. A filter on S is a nonempty family $\mathcal{F}$ of subsets of S that satisfies

- $\varnothing$ is not in $\mathcal{F}$,
- if A and B are in $\mathcal{F}$, then $A \cap B$ is in $\mathcal{F}$,
- if A is in $\mathcal{F}$ and $A \subseteq B$, then B is in $\mathcal{F}$.

If, in addition,

- for each $A \subseteq S$, either $A \in \mathcal{F}$ or $S - A \in \mathcal{F}$,

then $\mathcal{F}$ is an ultrafilter. An ultrafilter is a filter that is maximal for inclusion. For example, the family of all cofinite subsets of S (i.e. those subsets of S whose complements are finite) is a filter on S. The family of all subsets of S that contain a given element s of S is an ultrafilter on S and is said to be principal. An ultrafilter is principle as soon it contains a finite set. An ultrafilter that is not principle is said to be free. The intersection $\cap_{\mathcal{F}} A$ of a free ultrafilter $\mathcal{F}$ is the empty set.

A family $\mathcal{F}$ of subsets of S satisfies the finite intersection property if $A_1, A_2, \ldots, A_n \in \mathcal{F}_0$ implies $A_1 \cap A_2 \cap \ldots \cap A_n \in \mathcal{F}_0$. If one adds to $\mathcal{F}_0$ all the sets $B \subseteq S$ that contain finite intersections $A_1 \cap A_2 \cap \ldots \cap A_n$ of elements of $\mathcal{F}_0$, then one obtains a filter $\mathcal{F}_1$. By Zorn's lemma (which is equivalent to AC) there exists a maximal filter $\mathcal{F}$ on S that includes $\mathcal{F}_1$. This maximal filter $\mathcal{F}$ is an ultrafilter on S.

We provide some comments on AC and its relationship to the existence of a free ultrafilter. AC postulates for each family $\mathcal{D}$ of nonempty sets the existence of a function f such that $f(S) \in S$ for each set S in the family $\mathcal{D}$. The ultrafilter theorem states that—for each set S—each filter on S can be extended to an ultrafilter on S. The proof of the ultrafilter theorem uses AC (combine the fact that an ultrafilter is a filter that is maximal for inclusion and the equivalence of Zorn's lemma and AC). However, it is known that the ultrafilter theorem is weaker than AC (Halpern, 1967; Jech, 1978, p148). The ultrafilter theorem does not imply AC. Furthermore, the ultrafilter theorem is equivalent to the compactness theorem in first order theory (e.g. Chang and Keisler, 1992, p33).[5] This compactness theorem states the existence of a model of a set Σ of first order sentences as soon each finite subset of Σ has a model. The compactness theorem implies the order extension principle (also known as Szpilrajn's lemma). The argument is as follows. Let $(S, \leq)$ be a partially ordered set. Consider the first order language that contains constants for each s in S and a binary relation symbol $\precsim$. Let Σ be the following set of sentences:

- for each s, t, and u, we have $s \precsim t$ and $t \precsim u$ implies $s \precsim u$,
- for each s and t, we have $s \precsim t$ or $t \precsim s$ (or both),
- for each s and t such that $s \leq t$, we have $s \precsim t$,
- for each s and t such that $s \leq t$ and not $t \leq s$, we have $s \precsim t$ and not $t \precsim s$.

Induction on its number of elements implies that each finite subset of Σ has a model. Now, apply the compactness theorem. The existence of a complete ordering in S that extends the relation $\leq$ follows. End of the proof.
Hence, although Szpilrajn (1930) uses AC or Zorn's lemma, the full power of AC is not needed. The ultrafilter theorem or the equivalent compactness theorem is sufficient to obtain Szpilrajn's result. Let us visualize the various implications and equivalences:

$$\begin{array}{c} \text{AC} \\ \Updownarrow \\ \text{Zorn's lemma} \end{array} \quad \Longrightarrow \quad \begin{array}{c} \text{ultrafilter theorem} \\ \Updownarrow \\ \text{compactness theorem} \end{array} \quad \Longrightarrow \quad \text{Szpilrajn's lemma.}$$

[5] Chang and Keisler (1992, p219) provide an ultraproduct version of the compactness theorem and clearly reveal the link between the compactness theorem and the ultrafilter theorem.

As already mentioned, the ordering extension principle (Szpilrajn's lemma) has a nonconstructive component. The axiom of choice for families of finite sets (ACF) is weaker than the ordering principle (Jech, 1973). ACF postulates for each set U and each family $\mathcal{D}$ of pairwise disjoint nonempty subsets of U the existence of a function f such that $f(S) \in S$ for each S in the family $\mathcal{D}$.

The remainder of this section shows how to use a free ultrafilter on $\mathbb{N}_0$ to generate a complete SWR in the domain X. Let $\mathcal{F}$ be a free ultrafilter on $\mathbb{N}_0$. For each t in $\mathbb{N}_0$, let the relation $\precsim_t$ in Y^t be transitive, complete, Paretian, and (finite) anonymous. Define the relation $\precsim_{\mathcal{F}}$ in X as follows. For each x and y in X, we have

$$x \precsim_{\mathcal{F}} y \quad \text{if and only if} \quad \left\{ t \in \mathbb{N}_0 \;\middle|\; (x_1, x_2, \ldots, x_t) \preceq_t (y_1, y_2, \ldots, y_t) \right\} \in \mathcal{F}.$$

This relation $\precsim_{\mathcal{F}}$ is reflexive (use $\mathbb{N}_0 \in \mathcal{F}$), transitive (use the intersection property: if A and B are in $\mathcal{F}$, then $A \cap B \in \mathcal{F}$), complete (use the maximality principle: for each set $A \subseteq \mathbb{N}_0$, either $A \in \mathcal{F}$ or $\mathbb{N}_0 - A \in \mathcal{F}$), Paretian and finite anonymous (use the fact that each cofinite set belongs to $\mathcal{F}$). This technique of generating SWRs on X has been used by Fleurbaey and Michel (2003) and by Fleurbaey (2005). They argue that the above route is "more explicit" about the structure of $\precsim_{\mathcal{F}}$ than the use of Szpilrajn's lemma (with its proof based upon AC). As the ultrafilter theorem is weaker than AC, their observation indeed makes sense. Section 8 elaborates this discussion.

Starting from the sequence of the utilitarian (resp. leximin) SWRs on finite streams, one obtains an infinite version of the utilitarian ordering (resp. leximin) ordering that combines completeness, Pareto, and finite anonymity.

Given the family

$$\left\{ \precsim_{\mathcal{F}} \;\middle|\; \mathcal{F} \text{ a free ultrafilter on } \mathbb{N}_0 \right\}$$

of complete orderings new SWRs in the set of infinite utility streams can be obtained.

For example, define a relation R by $(x, y) \in R$ if either $x \prec_{\mathcal{F}_1} y$ or ($x \sim_{\mathcal{F}_1} y$ and $x \precsim_{\mathcal{F}_2} y$). This relation keeps the strict part $\prec_{\mathcal{F}_1}$ and narrows down the indifference sets through the superposition (in a lexicographical way) of the relation $\precsim_{\mathcal{F}_2}$. Extending this idea to a transfinite sequence of all free ultrafilters results in a relation for which only finite permutations are permissible.

Also, one can use a free ultrafilter $\mathbb{F}$ on the family of all free ultrafilters on $\mathbb{N}_0$ and define the relation $\precsim_{\mathbb{F}}$. For each x and y in X, we have

$$x \precsim_{\mathbb{F}} y \quad \text{if and only if} \quad \left\{ \mathcal{F} \;\middle|\; x \precsim_{\mathcal{F}} y \right\} \in \mathbb{F}.$$

All these complete orderings on the set of infinite utility streams involve the notion of a free ultrafilter. As the existence of a free ultrafilter depends upon AC, they have no explicit description.

4 Filters on the lattice of partitions

The notion of an ultrafilter on sets extends to an ultrafilter on a lattice of partitions. We follow Halbeisen and Löwe (2001) and recall the definitions and some results.

A partition of $\mathbb{N}_0$ is a family of pairwise disjoint nonempty sets such that their union coincides with $\mathbb{N}_0$. If A and B are two partitions of $\mathbb{N}_0$, we say that A is coarser than B (or that B is finer than A) and we write $A \sqsubseteq B$ if each piece in A is a union of pieces of B. The coarsest partition of $\mathbb{N}_0$ (everything in one piece) is denoted by $0 = \{\mathbb{N}_0\}$, the finest partition (all pieces of which are singletons) by 1. Each partition is in between 0 and 1.

We focus on the class Ω_0 of those partitions of $\mathbb{N}_0$ that consist out of infinitely many finite pieces. Partitions containing one (or more) infinite piece(s) are not distinguished, they are denoted by 0. We endow the class $\Omega = \Omega_0 \cup \{0\}$ with two operations $\cup$ and $\cap$. The partition $A \cup B$ is the coarsest partition in Ω that refines A and B, and the partition $A \cap B$ is the finest partition in Ω that is coarser than A and B. In case the partition $A \cap B$ contains an infinite piece, we put $A \cap B$ equal to 0. As such $(\Omega, \sqsubseteq)$ is a lattice.

A filter on the lattice $(\Omega, \sqsubseteq)$ is a collection $\mathcal{F}$ of members of Ω that satisfies

- 0 is not in $\mathcal{F}$,
- if both A and B are in $\mathcal{F}$, then $A \cap B$ is in $\mathcal{F}$,
- if B is in $\mathcal{F}$ and $B \sqsubseteq A$ (with A in Ω), then A is in $\mathcal{F}$.

A family $\mathcal{B} \subseteq \Omega$ is said to be a filter base if (i) $0 \notin \mathcal{B}$, and (ii) for each A_1 and A_2 in $\mathcal{B}$, there is a B in $\mathcal{B}$ such that $B \sqsubseteq A_1 \cap A_2$. In case $\mathcal{B}$ is a filter base, then the family $\mathcal{B}^+ = \{A \in \Omega |$ there is a B in $\mathcal{B}$ such that $B \sqsubseteq A\}$ is a filter on the lattice $(\Omega, \sqsubseteq)$. The filter $\mathcal{B}^+$ coincides with the intersection of all filters that include $\mathcal{B}$.

A filter that is maximal for inclusion is said to be an ultrafilter. Each ultrafilter $\mathcal{F}$ on $(\Omega, \sqsubseteq)$ is free, i.e. $\bigcap \{ A \mid A \in \mathcal{F} \} = 0$. We recall two facts on ultrafilters (Facts 2.1-2 in Halbeisen and Löwe, 2001, p321).

- A family $\mathcal{F}$ is an ultrafilter on $(\Omega, \sqsubseteq)$ if and only if for each A in Ω either $A \in \mathcal{F}$ or there is a B in $\mathcal{F}$ such that $A \cap B = 0$ (the 'either-or' being exclusive).
- If F is a family of elements of Ω with the finite intersection property (for each finite subfamily $\{ A_1, A_2, \ldots, A_n \} \subseteq F$ we have $A_1 \cap A_2 \cap \ldots \cap A_n \neq 0$), then there is an ultrafilter $\mathcal{F}$ on $(\Omega, \sqsubseteq)$ with $F \subseteq \mathcal{F}$.

The second fact is implied by Zorn's lemma. Hence, similar to the existence of a free ultrafilter on a set, also the existence of an ultrafilter on the lattice of partitions involves the use of AC in set theory. The notion "ultrafilter on a lattice" generalizes the notion "free ultrafilter on a set".

Example 1. Each infinite subset $C = \{n_1, n_2, \ldots, n_k, \ldots\}$ of $\mathbb{N}_0$ induces a partition P_C in Ω as follows:

$$P_C \;=\; \Big\{ [[1, n_1]]\,, [[n_1 + 1, n_2]]\,, \ldots, [[n_k + 1, n_{k+1}]]\,, \ldots \Big\},$$

where $[[i, j]]$ with $i \leq j$ is a shorthand for the set $\{i, i+1, \ldots, j-1, j\} \subset \mathbb{N}_0$. Now, let $\mathcal{F}_{\mathbb{N}_0}$ be a free filter on the set $\mathbb{N}_0$. Then, the family

$$\mathcal{F}_\Omega \;=\; \Big\{ P_C \in \Omega \,\Big|\, C \in \mathcal{F}_{\mathbb{N}_0} \Big\}$$

is a filter on the lattice $(\Omega, \sqsubseteq)$.
Moreover, $\mathcal{F}_\Omega$ is an ultrafilter on the lattice $(\Omega, \sqsubseteq)$ if and only if $\mathcal{F}_{\mathbb{N}_0}$ is a free ultrafilter on the set $\mathbb{N}_0$).

5 Ranking subsets of $\mathbb{N}_0$

In order to find a maximal group $\mathcal{Q}$ of permutations such that $\mathcal{Q}$-anonymity does not conflict with Pareto, we focus on the restriction of a SWR to $\mathcal{S}$—recall the assumption $\mathcal{S} \subset X$. Insight in how the SWR ranks different subsets of $\mathbb{N}_0$ provides insights in the strength of the Pareto axiom and the anonymity axiom.

The domain $[0, 1]^n$ of utility streams of finite length provides an analogue. Here, the Pareto axiom has a power equal to 2^{1-n}: the probability that the Pareto axiom is able to rank two randomly selected (uniform distribution) utility streams is equal to 2^{n-1}. The combination of anonymity and Pareto has a power equal to $2/(n + 1)$.
However, when restricted to the set $\mathcal{S}_n$ of vectors consisting out of zeros and ones,[6] the combination of anonymity and Pareto generates a complete ranking. Indeed, the combination of anonymity and Pareto boils down to counting the number of ones in each vector in $\mathcal{S}_n$, the higher the result the higher the vector is ranked. As such, the restrictions of the utilitarian, the leximin ordering, or any other ordering that combines Pareto and anonymity, to the set $\mathcal{S}_n$ coincide (with the counting procedure). In the finite setting, Pareto and anonymity characterize a complete ordering in the set of subsets of $\mathcal{S}_n$.

This note tries to extend this counting procedure on $\mathcal{S}_n$ to the set $\mathcal{S}$ of infinite sequences consisting out of zeros and ones. Similar to the finite setting, we take the conservative position that the Pareto axiom is the only motivation to judge one set better (bigger) than a second one. In order to expand the Pareto ordering one has to judge different sets as being equally large. In this approach, these indifferences are captured through some anonymity condition. From Corollary 1 we know that in order to avoid conflicts with Pareto only cyclic permutations are allowed.

[6] Again, each vector x in $\{0, 1\}^n$ is identified with the subset $\{\, t \,|\, x_t = 1 \,\}$ of $\{1, 2, \ldots, n\}$.

As a consequence we only consider a particular case of the conjecture of Fleurbaey and Michel. We look whether the extension of the Pareto ordering in $\mathcal{S}$ towards a complete ordering through the imposition of an anonymity axiom is possible without using AC (or weaker nonconstructive axioms). In this approach we do not examine, for instance, the lexicographical superposition of different orderings. Section 9 tackles the full conjecture.

6 Maximal anonymity

In this section we prove the first result. We start with some additional notation. Let the partition $A = \{ N_1, N_2, \ldots, N_k, \ldots \}$ belong to Ω_0. We will refer to

$$\mathrm{Sym}(A) \; = \; \mathrm{Sym}(N_1) \times \mathrm{Sym}(N_2) \times \cdots \times \mathrm{Sym}(N_k) \times \cdots,$$

with $\mathrm{Sym}(N_k)$ the group of all permutations on the finite set N_k, as the symmetric group of the partition A. The group $\mathrm{Sym}(A)$ stabilizes the partition A, i.e. this group collects all the permutations with an induced partition that is equal to or finer than A. We shorten $\mathrm{Sym}(\mathrm{Part}(\pi))$ to $\mathrm{Sym}(\pi)$. A group $\mathcal{Q}$ of permutations that includes $\mathrm{Sym}(\pi)$ for each π in $\mathcal{Q}$ is said to be a partition group. Furthermore, we write $\precsim_A$ for the relation $\precsim_{\mathrm{Sym}(A)}$; and we write $\precsim_\pi$ for the relation $\precsim_{\mathrm{Sym}(\pi)}$.

Let π belong to a partition group $\mathcal{Q}$ of cyclic permutations, then $\mathcal{Q}$-anonymity imposes indifference between a utility stream x, the permuted stream $y = x \circ \pi$, and all the streams obtained from x through rearrangements within the cycles of π.

For example, the partition group $\mathrm{Sym}(\pi_1)$ contains each permutation of the form

$$(1,2)^{k_1}\,(3,4)^{k_2}\,\cdots(2n-1,2n)^{k_n}\,\cdots$$

with k_i either 1 or 0 (where $(a,b)^1 = (a,b)$ and $(a,b)^0 = (a)(b)$). Therefore, if we impose $\mathrm{Sym}(\pi_1)$-anonymity, then the utility streams

$$x = \underbrace{1,0}, \underbrace{1,0}, \ldots, \underbrace{1,0}, \ldots \qquad \text{and} \qquad y = \underbrace{0,1}, \underbrace{0,1}, \ldots, \underbrace{0,1}, \ldots,$$

become equally good. In addition, for each subset S of $\mathbb{N}_0$, the utility stream x_S obtained from x by switching the utilities in two subsequent positions $2n-1$ and $2n$ for each n in S, is equally good as x (and y). The move from x to x_S involves 'less' switches than the move from x towards y. In this case, indifference between x and x_S can be interpreted as a 'weaker' demand than indifference between x and y. The next lemma indicates that partition groups allow us to shift the focus from permutations towards partitions.

Lemma 2. Let σ_A and σ_B be two cyclic permutations on $\mathbb{N}_0$. Then, $\mathrm{Sym}(\sigma_B)$ contains a permutation ρ such that $\rho \circ \sigma_A$ generates the partition $\mathrm{Part}(\sigma_A) \cap \mathrm{Part}(\sigma_B)$.

Proof. Denote $A = \mathrm{Part}(\sigma_A)$ and $B = \mathrm{Part}(\sigma_B)$. We prove the lemma in case $C = A \cap B$ consists out of an infinite number of finite sets. In case the partition C contains an infinite piece, the same ideas apply.

Without loss (otherwise re-enumerate $\mathbb{N}_0$), we assume the existence of an increasing sequence $n_1, n_2, \ldots, n_k, \ldots$ in $\mathbb{N}_0$ such that the partition C can be written as

$$C \;=\; \Big\{ \underbrace{[[1, n_1]]}_{S}, [[n_1+1, n_2]], \ldots , [[n_k+1, n_{k+1}]], \ldots \Big\}.$$

Both A and B are finer than C. We focus on one of the pieces in C, say $S = [[1, n_1]]$. Again, without loss, we assume that the restriction of σ_A to S is as follows

$$\sigma_A|_S \;=\; (1, 2, \ldots, k_1)(k_1+1, k_1+2, \ldots, k_2)\cdots(k_{m-1}+1, k_{m-1}+2, \ldots, n_1).$$

Denote the partition classes by $S_1 = [[1, k_1]], S_2 = [[k_1+1, k_2]], \ldots, S_m = [[k_{m-1}+1, n_1]]$. We construct a permutation ρ in $\mathrm{Sym}(B|_S)$ by induction. The partition $A \cap B$—when restricted to S—is equal to S. Hence, there exists a couple (ℓ_1, ℓ^1) in $S_1 \times (S - S_1)$ both belonging to one piece of B. Put $\rho(\ell_1) = (\ell^1)$. Let ℓ^1 belong to $S^1 = S_i$. Move on to the set $S_2 = S_1 \cup S^1$. Again, there exists a couple (ℓ_2, ℓ^2) in $S_2 \times (S - S_2)$ that both belong to one piece of B. Put $\rho(\ell_2) = \ell^2$. This procedure ends after m steps. Put the permutation ρ equal to $(\ell_1, \ell^1)(\ell_2, \ell^2)\cdots(\ell_m, \ell^m)$, elements of S that are not listed remain fixed.
The permutation $\rho \circ \sigma_A$ generates the cycle S in one piece. Repeat the whole construction for the other pieces in C and past the corresponding permutations together to obtain the result. □

Note that, in general, only the relation $\mathrm{Part}(\sigma_1) \cap \mathrm{Part}(\sigma_2) \sqsubseteq \mathrm{Part}(\sigma_1 \circ \sigma_2)$ holds. For example, consider the following cyclic permutations:

$$\begin{aligned}
\sigma_1 &= (1)(2,3,5,6,7,4)(8,11,13,14,15,12,10,9)\underline{(16,19,21,22,23,20,18,17)}\cdots,\\
\sigma_2 &= (1,2,3)(4,8,10,11,7,5)(6)(9)(12,16,18,19,15,13)(14)(17)\underline{(20,24,26,27,23,21)(22)(25)}\cdots.
\end{aligned}$$

The representation continues by repeating the underlined cycles taking into account a shift of +8. Here, $\mathrm{Part}(\sigma_1) \cap \mathrm{Part}(\sigma_2) = \mathbb{N}_0$ while both compositions $\sigma_2 \circ \sigma_1$ and $\sigma_1 \circ \sigma_2$ are cyclic:

$$\begin{aligned}
&\sigma_2 \circ \sigma_1 = \pi_1 = (1,2)(3,4)(5,6)(7,8)\cdots, \text{ and}\\
&\sigma_1 \circ \sigma_2 = (1,3)(2,5)(4,11)(6,7)(8,9)(10,13)(12,19)(14,15)(16,17)(18,21)\cdots.
\end{aligned}$$

We continue with some further notation. Let $\mathcal{B}$ be a family of partitions in Ω. The set of all permutations that stabilize an element of $\mathcal{B}$ is denoted by

$$\mathcal{Q}_{\mathcal{B}} \;=\; \Big\{ \pi \;\Big|\; \text{there is a } B \text{ in } \mathcal{B} \text{ such that } B \sqsubseteq \mathrm{Part}(\pi) \Big\}.$$

If $\mathcal{B}$ is a filter base, then $\mathcal{Q}_{\mathcal{B}}$ and $\mathcal{Q}_{\mathcal{B}^+}$ coincide.

Proposition 2. Let $\mathcal{B}$ be a family of partitions in Ω. Then, $\mathcal{Q}_{\mathcal{B}}$ is a maximal group of cyclic permutations if and only if $\mathcal{B}^+$ is an ultrafilter.

Proof. The if-part. Let $\mathcal{B}^+$ be a filter. Then, $0 \notin \mathcal{B}$, and $\mathcal{Q}_\mathcal{B}$ only contains cyclic permutations. If π and ρ belong to $\mathcal{Q}_\mathcal{B}$, then $\mathrm{Part}(\pi) \cap \mathrm{Part}(\rho)$ belongs to $\mathcal{B}^+$. Hence, $\mathcal{Q}_\mathcal{B}$ is closed for composition. Next, observe that the partition induced by a permutation coincides with the partition induced by its inverse permutation. Therefore, $\mathcal{Q}_\mathcal{B}$ is a (partition) group of cyclic permutations.

Now, suppose that $\mathcal{B}^+$ is an ultrafilter. We have to show that $\mathcal{Q}_\mathcal{B}$ is maximal. Therefore, assume that the cyclic permutation π is not in $\mathcal{Q}_\mathcal{B}$. The induced partition $A = \mathrm{Part}(\pi)$ does not belong to the ultrafilter $\mathcal{B}^+$. Hence, there is a B in $\mathcal{B}^+$ such that $A \cap B = 0$. Lemma 2 implies the existence of a permutation in $\mathrm{Sym}(B)$ such that the composition with π induces the partition 0. This composed permutation has an infinite cycle. Therefore, the permutation π cannot be added to $\mathcal{Q}_\mathcal{F}$ to generate a larger group of cyclic permutations.

The only-if-part. Let $\mathcal{Q}_\mathcal{B}$ be a maximal subgroup of cyclic permutations. We have to show that $\mathcal{B}^+$ is an ultrafilter. Since only cyclic permutations are involved, $0 \notin \mathcal{B}$. Next, assume that the partition A is not in $\mathcal{B}^+$. A permutation π that induces A does not belong to $\mathcal{Q}_\mathcal{B}$. Since, the group $\mathcal{Q}_\mathcal{B}$ is maximal, there is a σ in $\mathcal{Q}_\mathcal{B}$ such that $\pi \circ \sigma$ is not cyclic. Conclude that $A \cap \mathrm{Part}(\sigma) \sqsubseteq \mathrm{Part}(\pi \circ \sigma) = 0$ with $\mathrm{Part}(\sigma)$ in $\mathcal{B}^+$. □

The next proposition modifies Proposition 1. The conditions imposed upon the group $\mathcal{Q}$ of cyclic permutations are stronger than in Proposition 1. On the other hand, the set of permissible permutations is now defined through subsets of $\mathbb{N}_0$ (rather than trough arbitrary utility streams in X). The set of permissible partitions is defined to be

$$\tilde{\Pi}(\precsim) \;=\; \Big\{ A \in \Omega \,\Big|\, \text{for each } \pi \text{ in } \mathrm{Sym}(A) \text{ and for each } S \text{ in } \mathcal{S} \text{ we have } \pi(S) \sim S \Big\}.$$

In case the SWR $\precsim_1$ is a subrelation to the Paretian SWR $\precsim_2$, then $\tilde{\Pi}(\precsim_1) \subseteq \tilde{\Pi}(\precsim_2)$. Furthermore, if $\mathcal{B}$ is a collection of partitions in Ω, then $\precsim_\mathcal{B}$ denotes the relation $\precsim_{\mathcal{Q}_\mathcal{B}}$; and we write $\mathcal{B}$-anonymity instead of $\mathcal{Q}_\mathcal{B}$-anonymity.

Proposition 3. Let the family $\mathcal{B}$ of partitions in Ω be a filter base. Then, the relation $\precsim_\mathcal{B}$ is reflexive, transitive, Paretian, and $\mathcal{B}$-anonymous. Furthermore, the set of permissible partitions coincides with the filter $\mathcal{B}^+$.

Proof. The conditions imposed upon $\mathcal{B}$ turn $\mathcal{Q}_\mathcal{B}$ into a partition group of cyclic permutations. This group $\mathcal{Q}_\mathcal{B}$ coincides with $\mathcal{Q}_{\mathcal{B}+}$. Copying the proof of Proposition 3 in Mitra and Basu (2006) we obtain that $\precsim_\mathcal{B}$ is reflexive, transitive, Paretian, and $\mathcal{B}$-anonymous.

Let us now verify that $\tilde{\Pi}(\precsim_\mathcal{B})$ coincides with $\mathcal{B}^+$. The inclusion $\mathcal{B}^+ \subseteq \tilde{\Pi}(\precsim_\mathcal{B})$ is immediate. In case $\mathcal{B}^+$ is an ultrafilter also the reverse inclusion holds (otherwise there exists a cyclic permutation π outside the group $\mathcal{Q}_\mathcal{B}$ that keeps the indifference relation; as $\mathcal{Q}_\mathcal{B}$ is maximal $\mathcal{Q}_\mathcal{B} \cup \{\pi\}$ generates noncyclic permutations and a contradiction is obtained).

There remains one single statement to be proved: the inclusion $\tilde{\Pi}(\precsim_\mathcal{B}) \subseteq \mathcal{B}^+$ under the assumption that $\mathcal{B}^+$ is not an ultrafilter.[7] We show this inclusion by contradiction and

[7] In their Proposition 3, Mitra and Basu show the corresponding inclusion by means of the utility stream $\bar{x} = (1/2, 1/4, \ldots, 1/2^n, \ldots)$ which has a total $\Sigma_{i=1}^{\infty} \bar{x}_i = 1$. As we want to focus on relations in $\mathcal{S}$, we provide an alternative proof; cf. the alternative proof in Lemma 1.

assume $A \notin \mathcal{B}^+$. There exists an ultrafilter $\mathcal{F}$ that extends $\mathcal{B}$ and does not contain A (in the family $\mathcal{A}$ of all filters which do not contain A each chain has a maximal element, so by Zorn's lemma $\mathcal{A}$ has a maximal element that appears to be an ultrafilter). The relation $\precsim_{\mathcal{B}}$ is a subrelation to $\precsim_{\mathcal{F}}$, and $A \notin \tilde{\Pi}(\precsim_{\mathcal{F}})$. Hence, A does not belong to $\tilde{\Pi}(\precsim_{\mathcal{B}})$. □

Theorem 1. Let $\mathcal{B}$ be a family of subsets in Ω. If the relation $\precsim_{\mathcal{B}}$ in $\mathcal{S}$ is transitive, complete, Paretian, and finite anonymous, then $\mathcal{B}^+$ is an ultrafilter on $(\Omega, \sqsubseteq)$.

Proof. Let the relation $\precsim_{\mathcal{B}}$ satisfy each of the properties listed. Then, the collection $\mathcal{B}^+$ contains the partition 1 (reflexivity), is closed for intersection (transitivity), does not contain the partition 0 (Pareto). Hence, $\mathcal{B}^+$ is a filter and $\mathcal{Q}_{\mathcal{B}}$ is a partition group of cyclic permutations.

As $\precsim_{\mathcal{B}}$ is complete, the group $\mathcal{Q}_{\mathcal{B}}$ is maximal in $\mathcal{P}$. Indeed, otherwise there is a (partition) group $\mathcal{Q}$ such that $\mathcal{Q}_{\mathcal{B}} \subset \mathcal{Q} \subset \mathcal{P}$. The relation $\precsim_{\mathcal{Q}}$ is reflexive, transitive, Paretian, finite anonymous, and strictly extends the relation $\precsim_{\mathcal{B}}$ (here, we use the statement on permissible partitions in Proposition 3). As the relation $\precsim_{\mathcal{B}}$ is assumed to be complete we arrive at a contradiction. Hence, the group $\mathcal{Q}_{\mathcal{B}}$ is indeed maximal. Proposition 2 implies that $\mathcal{B}^+$ is an ultrafilter. □

Theorem 1 tackles a particular aspect of the conjecture of Fleurbaey and Michel. Consider a SWR in a set X of infinite utility streams such that its restriction to the set $\mathcal{S}$ is generated by the Pareto axiom in combination with an anonymity axiom (based upon a partition group). Then, this SWR either is incomplete or it generates a free ultrafilter on the lattice $(\Omega, \sqsubseteq)$ in which case there is no explicit description available.

7 Fixed and variable step permutations

Mitra and Basu (2006, Section 5) argue in favor of a particular group of cyclic permutations, to wit, the group of fixed step permutations. A permutation π is said to be fixed step if there is an n in $\mathbb{N}_0$ such that

$$F_n \;=\; \Big\{ [[1,n]]\,, [[n+1,2n]]\,, \ldots\,, [[kn+1,(k+1)n]]\,, \ldots \Big\} \;\sqsubseteq\; \mathrm{Part}(\pi).$$

Partitions of the form F_n are said to be fixed step. Let $\mathcal{F}_{\text{fix}}$ denote the filter on $(\Omega, \sqsubseteq)$ generated by the collection of fixed step partitions. The group $\mathcal{Q}_{\text{fix}}$ of fixed step permutations is a partition group.[8] Let $\mathcal{N}_{\text{fix}}$ be the filter on the set $\mathbb{N}_0$ generated by subsets of the form $\{n, 2n, \ldots, kn, \ldots\}$. The filter $\mathcal{N}_{\text{fix}}$ on the set $\mathbb{N}_0$ induces the filter $\mathcal{F}_{\text{fix}}$ on the lattice $(\Omega, \sqsubseteq)$. The main result in this section states a 1-1-relation between ultrafilters on the set $\mathbb{N}_0$ that extend $\mathcal{N}_{\text{fix}}$ and a particular class of complete, Paretian, and fixed step anonymous SWRs in $\mathcal{S}$.[9]

[8]Fixed step permutations have been proposed by Lauwers (1997) and by Fleurbaey and Michel (2003).
[9]This result also occurs in Fleurbaey (2006).

The filter $\mathcal{F}_{\text{fix}}$ is not an ultrafilter on $(\Omega, \sqsubseteq)$ and the corresponding relation $\precsim_{\text{fix}}$ (a shorthand for $\precsim_{\mathcal{Q}_{\text{fix}}}$) on the set $\mathcal{S}$ is not complete. For example, $\precsim_{\text{fix}}$ is unable to compare the sets $\{1, 6, 15, 28, \ldots\}$ and $\{3, 10, 21, 36, \ldots\}$.

In order to expand $\precsim_{\text{fix}}$ towards a complete relation we have to expand the filter $\mathcal{F}_{\text{fix}}$ towards an ultrafilter on $(\Omega, \sqsubseteq)$. Here, we propose to expand $\mathcal{N}_{\text{fix}}$ towards an ultrafilter on $\mathbb{N}_0$. In other words, we add partitions of the form

$$\Big\{ [[1, n_1]], [[n_1 + 1, n_2]], \ldots, [[n_k + 1, n_{k+1}]], \ldots \Big\},$$

with $n_1, n_2, \ldots, n_k, \ldots$ an increasing sequence in $\mathbb{N}_0$. Such a partition is said to be a variable step partition. A permutation that generates a variable step partition is said to be variable step. Let the set $\mathcal{Q}_{\text{var}}$ collect all variable step permutations. Fixed step permutations are variable step and variable step permutations are cyclic, $\mathcal{Q}_{\text{fix}} \subset \mathcal{Q}_{\text{var}} \subset \mathcal{P}$. On the other hand, the next lemma reveals that $\mathcal{Q}_{\text{var}}$ is not a group. Nevertheless, Lemmas 3 and 4 provide further arguments in favor of the set of variable step permutations.

Lemma 3. The set $\mathcal{Q}_{\text{var}}$ of variable step permutations generates the group $\text{Sym}(\mathbb{N}_0)$ of all permutations on $\mathbb{N}_0$. In particular, each permutation in $\mathbb{N}_0$ can be decomposed into two variable step permutations.

Proof. Let $\pi : \mathbb{N}_0 \to \mathbb{N}_0$ be a permutation. We construct two variable step permutations σ and τ such that $\pi = \sigma \circ \tau$. The construction is done via subsequent extensions of two permutations on finite sets of increasing length.

Put $\sigma(1) = \pi(1)$ and $\tau(1) = 1$. Let $t_1 > \pi(1)$ and extend σ to a permutation on the set $T_1 = [[1, t_1]]$. Define τ on the set T_1 such that $\sigma \circ \tau$ coincides with π (when restricted to the domain of the currently defined composition $\sigma \circ \tau$). Let $t_2 > \max \{\rho^{-1}(T_1) \cup \tau(T_1)\}$ and extend τ to a permutation on the set $T_2 = [[1, t_2]]$. Then extend σ to the set T_2 such that $\sigma \circ \tau$ coincides with π. Let $t_3 > \max \sigma(T_2)$ and extend σ to a permutation on the set $T_3 = [[1, t_3]]$, and so forth.

The permutation σ satisfies $\sigma(T_{2k+1}) = T_{2k+1}$ for each k in $\mathbb{N}_0$. Similarly, the permutation τ satisfies $\tau(1) = 1$ and $\tau(T_{2k}) = T_{2k}$ for each k in $\mathbb{N}_0$. Therefore the permutations σ and τ both belong to $\mathcal{Q}_{\text{var}}$. □

Hence, combining variable step anonymity and transitivity upon a relation implies the imposition of $\text{Sym}(\mathbb{N}_0)$-anonymity. To illustrate this lemma further, we mention that the infinite cycle π_3 coincides with $\pi_1 \circ \pi_2$. Therefore, a partition group of cyclic permutations cannot contain both π_1 and π_2. Each anonymity condition based upon a group of variable step permutations, strong enough to generate a complete ranking on $\mathcal{S}$, and weak enough to allow for Pareto, should impose either π_1-anonymity or π_2-anonymity (the either-or being exclusive). Next, we show that each cyclic permutation can be rewritten (after a re-numbering of $\mathbb{N}_0$) as a variable step permutation.

Lemma 4. Each cyclic permutation is conjugated to a variable step permutation.

Proof. Let π be in $\mathcal{P}$. We have to find a permutation $\tilde{\pi}$ in $\mathcal{Q}_{\text{var}}$ and a permutation σ of the set $\mathbb{N}_0$, such that $\pi = \sigma^{-1} \circ \tilde{\pi} \circ \sigma$. Present π as a product of cycles:

$$\pi \; = \; (a_{11}, \ldots, a_{1k_1})(a_{21}, \ldots, a_{2k_2}) \cdots (a_{n1}, \ldots, a_{nk_n}) \cdots.$$

Then, it suffices to check the identity $\tilde{\pi} = \sigma \circ \pi \circ \sigma^{-1}$ with

$$\tilde{\pi} \; = \; (1, \ldots, k_1)(k_1 + 1, \ldots, k_2) \cdots (k_{n-1} + 1, \ldots, k_n) \cdots,$$

and $\sigma : a_{ij} \mapsto k_{i-1} + j$ with $i = 1, 2, \ldots$ and $j = 1, 2, \ldots, k_i - k_{i-1}$. We put $k_0 = 0$. □

Both lemmas attract the focus upon extensions of the group $\mathcal{Q}_{\text{fix}}$ within the set $\mathcal{Q}_{\text{var}}$. In order to extend the partition group $\mathcal{Q}_{\text{fix}}$ to a maximal group of cyclic permutations, one can extend the filter $\mathcal{F}_{\text{fix}}$ to an ultrafilter $\mathcal{F}$ on the lattice $(\Omega, \sqsubseteq)$. Here, we propose to extend the filter $\mathcal{N}_{\text{fix}}$ to an ultrafilter $\mathcal{N}$ on the set $\mathbb{N}_0$. We reformulate the results of the previous section in terms of filters on $\mathbb{N}_0$.

For a family $\mathcal{B}$ of subsets of $\mathbb{N}_0$, the relation $\precsim_{\mathcal{B}}$ denotes the relation $\precsim_{\mathcal{B}_\Omega}$, with $\mathcal{B}_\Omega$ the family of partitions in Ω induced by the elements in $\mathcal{B}$. Furthermore, $\mathcal{B}^+$ is the filter on the set $\mathbb{N}_0$ generated by $\mathcal{B}$, i.e. $\mathcal{B}^+$ is the intersection of all filters that include $\mathcal{B}$.

Corollary 3.[10] Let $\mathcal{B}$ be a family of subsets of $\mathbb{N}_0$. If the relation $\precsim_{\mathcal{B}}$ is reflexive, transitive, Paretian, finite anonymous, and complete; then $\mathcal{B}^+$ is a free ultrafilter on the set $\mathbb{N}_0$.

Recall the definition of the relation $\precsim_{\mathcal{N}}$ in $\mathcal{S}$ with $\mathcal{N}$ a free ultrafilter on the set $\mathbb{N}_0$ (Section 3). For each S and T in $\mathcal{S}$ we have

$$S \precsim_{\mathcal{N}} T \quad \text{if and only if} \quad \Big\{ t \in \mathbb{N}_0 \;\Big|\; |\, S \cap \{1, 2, \ldots, t\} \,| \; \leq \; |\, T \cap \{1, 2, \ldots, t\} \,| \Big\} \in \mathcal{N}.$$ [11]

This relation is reflexive, transitive, Paretian, finite anonymous, and complete. Hence, the reverse statement in Corollary 3 also holds.

Proposition 4. Let $\mathcal{B}$ be a family of subsets of $\mathbb{N}_0$. Then, the relation $\precsim_{\mathcal{B}}$ is transitive, complete, Paretian, and finite anonymous if and only if $\mathcal{B}^+$ is a free ultrafilter on $\mathbb{N}_0$.

Proposition 4 reveals that—in the setting of filters on $\mathbb{N}_0$—the maximal amount of anonymity respecting the axiom of Pareto is large enough to generate a complete SWR.

8 Overtaking, Svensson, and a non-standard SWF

We recall the overtaking criterion and Svennson's (1980) extension. We argue that the ultrafilter approach performs better. Furthermore, the ultrafilter-based criterion is representable by a non-standard social welfare function. We close this section by observing a weak form of impatience.

[10]Marc Fleurbaey (2006) establishes the result mentioned in this corollary.

[11]The cardinality of a set S is denoted by $|S|$.

Von Weizsäcker (1965) and Atsumi (1965) define the following overtaking criterion. For each x and y in $\mathbb{R}^{\mathbb{N}_0}$, we have

$$x \precsim_V y \quad \text{if and only if there is a } T \text{ in } \mathbb{N}_0 \text{ such that } \Sigma_{i=1}^{t}(y_i - x_i) \geq 0 \text{ for each } t \geq T.$$

Svennson (1980) extends this criterion. First, the set of utility streams is partitioned through the equivalence relation $x \equiv y$ if and only if $\sum_1^n |x_k - y_k|$ converges when n goes to infinity. Then, AC is used to select one representative element from each equivalence class. Each representative element is used to rank the equivalence class it belongs to. Next, Szpilrajn's lemma is used to extend the overtaking criterion (when restricted to the set of representative elements) to a complete ordering in the set of representative elements. Although the resulting criterion is transitive, complete, Pareto, and finitely anonymous, its definition involves AC to select a set of representatives and Szpilrajn's lemma (with its AC-based proof) to extend a partial ordering.

Free ultrafilters provide an alternative route. Let $\mathcal{N}_{\text{fix}}$ be the (constructible) filter on $\mathbb{N}_0$ that induces the filter $\mathcal{F}_{\text{fix}}$ of fixed step partitions. Define the partial ordering $\precsim_{\text{fix}}$: for each pair x and y of infinite utility streams, we have

$$x \precsim_{\text{fix}} y \quad \text{if and only if} \quad \left\{ t \;\middle|\; \frac{x_1 + x_2 + \cdots + x_t}{t} \leq \frac{y_1 + y_2 + \cdots + y_t}{t} \right\} \in \mathcal{N}_{\text{fix}}.$$

Let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}_0$ that includes the filter $\mathcal{N}_{\text{fix}}$. The relation $\precsim_{\mathcal{F}}$ extends $\precsim_{\text{fix}}$ and is transitive, complete, Paretian, and finite anonymous. This ultrafilter approach exhibits at least two advantages over Svennson's approach. *First*, the constructible component $\precsim_{\text{fix}}$ is able to rank more utility streams than the constructible component of Svennson's criterion. The relation $\precsim_{\mathcal{F}}$ satisfies fixed step anonymity and judges the streams $x = (0,1,0,1,\ldots,0,1,\ldots)$ and $y = (1,0,1,0,\ldots,1,0,\ldots)$ equally good. This in contrast to Svennson who leaves the decision on x and y to AC. Indeed, the streams x and y are not equivalent: for each odd n we have $\sum_1^n |x_k - y_k| = 1$. *Second*, the ultrafilter-based SWR inherits each universal property of the truncated relations. A universal property is expressed by a first order sentence involving only universal quantifiers. Transitivity, completeness, negative transitivity, and n-acyclicity are examples. On the other hand, non-universal properties should be treated with care. For example, continuity involves an existential quantifier ($\forall \varepsilon \exists \delta \ldots$) and is lost: each finite utilitarian rule is continuous while the ultrafilter-utilitarian SWR is discontinuous. Conclusion: although the criterion $\precsim_{\mathcal{F}}$ still relies on a nonconstructive object, it performs better than Svennson's criterion.

Next, we observe that the SWR $\precsim_{\mathcal{F}}$ can be represented by a map from X to a non-standard model of the real numbers. The ultraproduct $\mathbb{R}^* = \mathbb{R}^{\mathbb{N}_0}/\mathcal{F}$ extends the set of real numbers with well defined infinitesimals and is rich enough to allow the following representation:

$$X \longrightarrow \mathbb{R}^* : x \longmapsto \Big(x_1, \frac{x_1 + x_2}{2}, \ldots, \frac{x_1 + x_2 + \cdots + x_t}{t}, \ldots \Big) \Big/ \mathcal{F}.$$

Due to the normalization (the truncated sum of length t is divided by t) this non-standard social welfare function maps the constant utility stream $(r, r, \ldots, r, \ldots)$ to r in $\mathbb{R} \subset \mathbb{R}^*$. Of course, this representation is equally practical relevant as the criterion $\precsim_{\mathcal{F}}$ itself.

The fixed step anonymous relation $\precsim_{\mathcal{F}}$ still suffers form impatience. For example, this criterion ranks $x = (1, 0, \ldots, 0, \ldots)$ strictly above $y = (1/2, 1/4, \ldots, 1/2^k, \ldots)$. Indeed, for each t the utilitarian criterion ranks the truncated vector $x^t = (1, 0, \ldots, 0)$ strictly above $y^t = (1/2, \ldots, 1/2^t)$. On the other hand, both x and y generate the very same total utility: $\Sigma x_i = \Sigma y_i = 1$. If utilitarians take the total utility as a base to rank different projects, then they should consider x and y equally good. In order to avoid this type of 'impatience' we add the following sum-principle (e.g. Lauwers and Vallentyne, 2004).

Definition 2. Let $\precsim$ be a SWR. Then, $\precsim$ is said to satisfy the sum-principle if for each x and y in $\mathbb{R}^{\mathbb{N}_0}$, the inequalities $0 \leq \sum_{i=1}^{\infty}(y_i - x_i) < \infty$ imply that $x \precsim y$.

We modify the above relation $\precsim_{\mathcal{F}}$ into a two-step procedure. In order to compare two utility streams x and y in $\mathbb{R}^{\mathbb{N}_0}$, we first check whether $s = \sum_{i=1}^{\infty}(y_i - x_i)$ belongs to $\mathbb{R}$. If the answer is yes, then x is better than, worse than, or equally good as y in case $s < 0$, $s > 0$, or $s = 0$. If the answer is no, then we rank x and y via the rule $\precsim_{\mathcal{F}}$. This two-step procedure generates a transitive, complete, Paretian, and fixed step anonymous relation in $\mathbb{R}^{\mathbb{N}_0}$. In addition, this procedure respects the sum-principle.

9 Non-Ramsey sets

This section develops the second result. We introduce some further notation, recall the Ramsey property, and prove Theorem 2. Next, we replace the finite anonymity condition by a finite insensitivity condition and show how completeness of the SWR either implies the existence of a non-measurable set or results in an almost null SWR (i.e. a relation that with probability one proposes indifference).

For each infinite set I, let $[I]^\infty$ collect all the infinite subsets of I. A finite partition $\mathcal{A}$ of $[\mathbb{N}_0]^\infty$ is said to be Ramsey in case there is an infinite subset M of $\mathbb{N}_0$ such that $[M]^\infty$ is included in one of the partition classes of $\mathcal{A}$. In this case the subset M is said to be homogenous for the partition $\mathcal{A}$. This partition property can be stated in terms of colorings. The partition $\mathcal{A}$ is the result of a coloring map $c : [I]^\infty \to C$ with C a set of different colorings; the cardinality of C coincides with the (finite) number of partition classes in $\mathcal{A}$. Ramsey's property, then, imposes the existence of an infinite set M for which $[M]^\infty$ is monochromatic. In the finite setting each partition of $[I]^n$—the set of subsets of I with exactly n elements—has the Ramsey property. In the infinite setting, however, there exist partitions (with two partition classes) that do not have the Ramsey property. Building upon the work of Solovay (1970), Mathias (1977) showed that the existence of Ramsey sets does not follow from the axioms of Zermelo-Fraenkel without AC.

We develop an extra piece of notation. Let $i < j$ be two natural numbers in $\mathbb{N}_0$. The notation $[[i, j[[$ is a shorthand for the set $\{i, i+1, \ldots, j-1\}$. We present the infinite set $S = \{x_1, x_2, \ldots, x_k, \ldots\} \subseteq \mathbb{N}_0$ as an increasing sequence

$$x = (x_1, x_2, \ldots, x_k, \ldots),$$

with $x_k < x_{k+1}$ for each k. We connect two infinite sets to x:

$$x' = [[x_1, x_2[[\cup [[x_3, x_4[[\cup \ldots \cup [[x_{2k-1}, x_{2k}[[\cup \ldots,$$

and

$$x'' = [[x_2, x_3[[\cup [[x_4, x_5[[\cup \ldots \cup [[x_{2k}, x_{2k+1}[[\cup \ldots.$$

These triples (x, x', x'') are crucial in our main result.

Theorem 2. Each transitive, complete, intermediate Paretian, and finite anonymous relation in $[\mathbb{N}_0]^\infty$ generates a non-Ramsey set.

Proof. Let the relation $\precsim$ in $[\mathbb{N}_0]^\infty$ satisfy the four conditions. Define three sets:

$$A = \{x \mid x' \prec x''\}, \qquad B = \{x \mid x'' \prec x'\}, \qquad C = \{x \mid x' \sim x''\},$$

with x' and x'' as above. As the relation $\precsim$ is complete, the sets A, B, and C form a partition of $[\mathbb{N}_0]^\infty$. We now prove that for each infinite set $x = (x_1, x_2, \ldots, x_k, \ldots)$ in $[\mathbb{N}_0]^\infty$ there exists a subset y such that x and y belong to two different partition classes. We distinguish three cases: $x \in A$, $x \in B$, and $x \in C$.

Case 1. The set x is in A, in other words $x' \prec x''$.

Consider the subset $y = (x_2, x_3, \ldots, x_k, \ldots)$ obtained from x by cancelling out x_1. Then,

$$y'' = [[x_3, x_4[[\cup [[x_5, x_6[[\cup \ldots \cup [[x_{2k-1}, x_{2k}[[\cup \ldots,$$

and

$$y' = [[x_2, x_3[[\cup [[x_4, x_5[[\cup \ldots \cup [[x_{2k}, x_{2k+1}[[\cup \ldots.$$

Note that $y'' \cup [[x_1, x_2[[= x'$ and that $y' = x''$. From $x' \prec x''$ it follows that $y'' \prec y'$.

Case 2. The set x is in B, in other words $x'' \prec x'$.

Again we consider $y = (x_2, x_3, \ldots, x_k, \ldots)$. In case $y' \prec y''$ we are done. Otherwise, we further enlarge y'' by dropping elements in x. In particular, we drop x_4 and x_5, x_6 and x_7 until x_{2i} and x_{2i+1} to obtain the set $z = (x_2, x_3, x_{2i+2}, x_{2i+3}, \ldots, x_k, \ldots)$ such that:

$$z'' = [[x_3, x_{2i+2}[[\cup [[x_{2i+3}, x_{2i+4}[[\cup \ldots \cup [[x_{2k-1}, x_{2k}[[\cup \ldots,$$

and

$$z' = [[x_2, x_3[[\cup [[x_{2i+2}, x_{2i+4}[[\cup \ldots \cup [[x_{2k}, x_{2k+1}[[\cup \ldots.$$

Note that z'' includes $x' \setminus [[x_1, x_2[[$ and that $z' \subset x''$. The value of i should be large enough to guarantee that the cardinality of $S = [[x_3, x_{2i+2}[[$ exceeds the cardinality of

$T = [[x_1, x_2[[\cup[[x_3, x_4[[\cup \ldots \cup [[x_{2i+1}, x_{2i+2}[[$. By finite anonymity T can be embedded in S. The relation $z' \prec z''$ follows.

Case 3. The set x is in C, in other words $x' \sim x''$.
Drop the elements x_2 and x_3, x_6 and x_7, x_{10} and x_{11} and so forth (x_{2+4k} and x_{3+4k} for each $k = 0, 1, 2, \ldots$). We obtain from x the subset y that satisfies

$$y' \ = \ [[x_1, x_4[[\cup [[x_5, x_8[[\cup [[x_9, x_{12}[[\cup \ \ldots,$$

and

$$y'' \ = \ [[x_4, x_5[[\cup [[x_8, x_9[[\cup [[x_{12}, x_{13}[[\cup \ \ldots.$$

Note that y' includes x' and has infinitely many elements more than x'. Also x'' includes y'' and has infinitely many elements more than y''. Due to intermediate Pareto, we obtain $y'' \prec y'$.
Conclude that starting from an element x in A (B, or C) we are able to find subsets of x in different partition classes. The partitioning $\{A\, ,\, B \cup C\}$ of $[\mathbb{N}_0]^\infty$ does not have the Ramsey property. The set A is non-Ramsey. □

Theorem 2 confirms the full conjecture of Fleurbaey and Michel. Consider a SWR in a domain X such that its restriction to the set $[\mathbb{N}_0]^\infty$—i.e. the set of utility streams made up with zeros and infinitely many ones—is intermediate Paretian and finite anonymous. Then, this SWR either is incomplete or it generates a non-Ramsey set in which case there is no explicit description available.

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