Price Competition with Population Uncertainty

Klaus Ritzberger

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Abstract

The Bertrand paradox holds that price competition among at least two firms eliminates all profits in equilibrium, when firms have identical constant marginal costs. This assumes that the number of competitors is common knowledge among firms. If firms are uncertain about the number of their competitors, there is no pure strategy equilibrium. But in mixed strategies an equilibrium exists. In this equilibrium it takes a large market to wipe out profits. Thus, with population uncertainty, two are not enough for competition.

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1 Introduction

The Bertrand (1883) model of price competition on a market for a homogeneous product is one of the basic models of imperfect competition. Its prediction appears somewhat counter-intuitive, though: When firms have identical constant marginal costs, price equals marginal cost and equilibrium

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† Vienna Graduate School of Finance and Institute for Advanced Studies, Department of Economics and Finance, Stumpergasse 56, A-1060 Vienna, Austria. Tel. (+43-1) 599 91-153, Fax. (+43-1) 599 91-555, e-mail. ritzbe@ihs.ac.at
profits are zero. Since this is independent of the (given) number of firms, “two are enough for competition.”

More generally, this also holds, for all but the most efficient firm, if firms have different, but constant marginal costs. For, any price above the second-smallest marginal cost can be undercut by the most efficient firm. At any price below the second-smallest marginal cost the most efficient firm can slightly raise the price without risking a loss of customers. Therefore, in equilibrium the most and the second-most efficient firm will both ask the second-smallest marginal cost and consumers will endogenously decide to buy at the most efficient firm, as otherwise the latter would undercut.\footnote{This argument treats customers as players, whose behavior at indifference adjusts so as to support equilibrium; see Simon and Zame (1990). If the market split at ties is exogenous, mixed equilibria obtain; see Blume (2003). On the case of strictly convex cost functions see Thepot (1995) or Hoernig (2002).} Hence, the equilibrium markup is limited by the cost advantage of the most efficient firm. In particular, if at least two firms have the smallest marginal cost, equilibrium profits are again zero. This is in stark contrast to the other popular model of oligopolistic competition, the Cournot model, where equilibrium profits are always positive.

The “Bertrand paradox” is derived under the assumption that all relevant parameters, like cost and demand functions, are commonly known among the competitors. In particular, it assumes that the number of competitors in the relevant market is common knowledge among firms. With free entry this corresponds to a situation, where cost function for all potential entrants are commonly known, so that it is perfectly predictable who will enter. In this sense, “two are enough for competition, provided the number of active competitors is commonly known.”

The assumption of a given and commonly known number of firms on the market does not always seem plausible, though. It may be appropriate in a mature market with high barriers to entry, where all firms on the market have detailed information about their rivals. In emerging industries with high turnover rates and low entry and exit costs, on the other hand, there may be considerable uncertainty about competitors. If cost functions of potential entrants are uncertain or entry and exit costs are private information, for instance, the number of competitors is also uncertain. If firms cannot predict with certainty when their rivals will default, the perceived number of active competitors is uncertain again. So, there is reason to consider an oligopolistic markets, where the number of active firms is uncertain.
Interestingly, results change when there is uncertainty about the number of rivals. This paper embeds a generalized Bertrand model into a game with population uncertainty à la Myerson (1998, 2000). That is, a random number of firms is drawn to compete in prices on a market for a homogeneous commodity. Each firm learns whether or not it was drawn and what its cost function is. But firms do not observe the number of competitors, nor the costs of competitors.

With this set-up there is no equilibrium in pure strategies. A mixed equilibrium still exists, but it entails positive (expected) profits and price dispersion. In line with economic intuition, profits decline as the expected number of competitors grows. If the expected number of competitors diverges fast enough, as it is the case in some interesting special cases, profits are squeezed to zero and approximately competitive outcomes obtain in the limit.

The framework is general enough so as to encompass a number of interesting models as special cases. First, the distribution of the number of competitors is arbitrary. A deterministic number of rivals is, therefore, a special case, as is a finite maximum number of rivals. Second, firms may have different and random cost functions (“types”) that are weakly convex. Putting a deterministic number of firms together with randomly assigned cost functions yields a model of price competition under uncertainty about rivals’ costs, where costs may be strictly convex. This special case corresponds to the model by Spulber (1995), who already observed that uncertainty about cost functions of rival firms mitigates the Bertrand paradox. With two firms, two types, and linear costs this gives a stochastic variant of the model by Blume (2003). Third, as the distribution of types is arbitrary, it may be concentrated on a single cost function. Putting together a finite maximum number of rivals with a single type and constant marginal costs yields the model by Janssen and Rasmusen (2002). The combination of a deterministic number of rivals with a single type and constant marginal costs, of course, yields the classical Bertrand (1883) model. Finally, when the distribution of the number of competitors is Poisson (as suggested by Myerson, 1998, 2000), a model is obtained, where it can be shown that the distribution of prices approaches an atom at marginal cost pricing as the mean population size diverges.

\(^2\) For a more general model of games with population uncertainty see Milchtaich (2004).

\(^3\) Spulber (1995) allows also for fixed costs and a continuum of types, both of which are not included in the present model.
The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 states the results, and Section 4 discusses special cases. Section 5 summarizes.

2 The Model

There is a countably infinite pool of potentially active firms from which a random draw selects some to compete on the market. Each firm learns whether or not it is drawn, but not which and how many other firms are drawn. Moreover, each firm that has been drawn gets randomly assigned a type that determines its cost function. The cost function (type) is private information to active players.

Once on the market, firms compete for customers by (simultaneously) setting prices. Consumers observe all prices set by active firms and then decide to buy from the firms that offer the best price. As consumers care only about the price, but not about the identity of the firm from which they buy, and firms produce whatever quantity is ordered (without capacity constraints), the demand side of the market can be modelled by a demand function: a twice continuously differentiable function \( Q : \mathbb{R}_+ \to \mathbb{R}_+ \) that is strictly decreasing (when it is positive), weakly log-concave, and satisfies \( Q(p^*) = 0 \) for some finite \( p^* \in \mathbb{R}_+ \).

A firm’s cost depends on its output and its type. A type \( t = 1, \ldots, T \) gets randomly assigned to each firm (that has been drawn) according to a probability distribution \( \rho_n = (\rho_n(t))_{t=1,\ldots,T} \gg 0 \) independently across active firms, but possibly depending on the number \( n \) of firms drawn to play. (Hence, \( \rho_n(t) = \Pr(t|n) \) is the conditional probability that a firm gets assigned type \( t \) given that \( n \) firms are drawn.) The dependence of costs on output and the type is described by a family of twice continuously differentiable functions \( C_t : \mathbb{R}_+ \to \mathbb{R}_+ \) that are strictly increasing and (weakly) convex, satisfy \( C_t(0) = 0 \) and \( C'_t(0) \geq 0 \), and that both total costs \( C_t(q) \) and marginal costs \( C'_t(q) \) strictly decrease as the type \( t \) increases, for all \( q \in \mathbb{R}_+ \). Hence, higher \( t \) unambiguously corresponds to more efficiency in production.

Accordingly, the monopoly profit function of an active firm of type \( t = 1, \ldots, T \) is given by

\[ V_t(p) = p(Q(p) - C_t(Q(p))) \]

for all prices \( p \in \mathbb{R}_+ \). The assumptions on \( Q \) and \( C \) imply that \( V_t \) is a
quasi-concave function of $p$. For,

$$V'_t(p) = Q + pQ' - C'_tQ' = 0$$

$$\Rightarrow V''_t(p)|_{V'_t=0} = 2Q' + pQ'' - C''_t(Q')^2 - C'_tQ''$$

$$\leq Q' - C''_t(Q')^2 < 0$$

because $Q' < 0$, $C''_t \geq 0$, and log-concavity of $Q$ implies $QQ'' \leq (Q')^2$. Any extremum of $V_t$ is, therefore, a maximum. For, $V_t$ is continuous, $V'_t(0) > 0$, and $V'_t(p^*) < 0$, so that (2) has a solution by the intermediate value theorem and the desired conclusion follows. Moreover, that the second-order condition holds with strict inequality, implies that (2) has a unique solution.

For each type $t = 1, \ldots, T$ denote the solution to (2) by $p_t$, viz. the monopoly price where marginal revenue equals marginal cost.\footnote{If fixed costs are strictly smaller than the monopoly profit $V_t(p_t)$, they could be easily incorporated in the present model.} Because $V'_t$ is downward sloping at $p_t$ (by the second-order condition) and $C'_t$ decreases with $t$, the monopoly price $p_t$ is a strictly decreasing function of the type $t = 1, \ldots, T$, so $p_T < p_{T-1} < \ldots < p_1$. Furthermore, at any fixed price $p \in \mathbb{R}_+$ the profit $V_t(p)$ strictly increases with $t$, and the marginal profit $V'_t(p)$ strictly decreases with $t$.

To model the firms’ uncertainty about the number of rivals, let $\pi_t = (\pi_t(n))_{n=0,1,\ldots}$ denote the conditional probability distribution over the number of remaining active firms given that a firm of type $t$ has been drawn, i.e., $\pi_t(n) \geq 0$ for all $n = 0, 1, \ldots$ and $\sum_{n=0}^{\infty} \pi_t(n) = 1$. That is, $\pi_t$ captures the beliefs of a firm of type $t$ on the number of its competitors. Since the actual numbers of players of each type are generally uncertain, there is nothing beyond the type that a firm can base its beliefs on. But, because the type assignment is here assumed independent of the number of competitors, conditioning on the type does not change a firm’s forecast about the number of other firms in the market. Therefore, the subscript $t$ on beliefs can be dropped, i.e., $\pi_t = \pi = (\pi(n))_{n=0,1,\ldots}$ for all types $t = 1, \ldots, T$.

2.1 An Example

To fix ideas, consider a simple example with $T = 1$ and $\pi = (1 - \alpha, \alpha, 0, \ldots)$ for some $\alpha \in (0, 1)$, i.e., cost functions are common knowledge and there can be at most two firms on the market. Also assume that $Q(p) = \max \{0, 1 - p\}$.
and $C_1(q) = 0$, i.e., linear demand and constant (and identical) marginal costs normalized to zero. The monopoly price is $\bar{p}_1 = 1/2$. Clearly, in equilibrium no firm will set a price above the monopoly price.

Each of the two firms believes that by setting a price $p \in [0, 1/2]$ with probability $1 - \alpha$ she will obtain the monopoly profit $V_1(p) = p(1 - p)$ and that with probability $\alpha$ she will face an identical competitor. Therefore, in any equilibrium each firm must expect to earn at least $(1 - \alpha) V_1(\bar{p}_1) = (1 - \alpha)/4 > 0$, because this is what she can make by betting on the event that there will be no competition. Therefore, firms will set prices strictly above marginal costs in equilibrium, leaving room for undercutting. This implies that there can be no equilibrium in pure strategies.

For, if there were a pure strategy equilibrium, where one firm sets a strictly smaller price than the other, the former could gain by slightly increasing its price, because this would raise its profit in both events, without risking a loss of market share when both are active. If there were a pure strategy equilibrium, where both firms set the same price, the firm that obtains a market share of less than $1$ in the event that both are active could gain by slightly undercutting, since its market share when both are active would jump to $1$.

But there is a mixed strategy equilibrium. Define the function $\sigma : [0, 1] \to [0, 1]$ by

$$\sigma(p) = \max \left\{ 0, \frac{1}{\alpha} - \frac{1 - \alpha}{4 \alpha p (1 - p)} \right\}$$
for all \( p \in [0, 1/2] \) and by \( \sigma (p) = 1 \) for \( p > 1/2 \) (see Figure 1). We claim that both firms mixing according to the cumulative distribution function \( \sigma \) constitutes an equilibrium. For, if the opponent mixes according to \( \sigma \) a firm expects to earn

\[
p (1 - p) [1 - \alpha + \alpha (1 - \sigma (p))] = \frac{1 - \alpha}{4} > 0
\]

by setting any price \( p \in [1/2 - \sqrt{\alpha}/2, 1/2] \), less than \( (1 - \alpha)/4 \) by setting any price above \( 1/2 \), and \( p (1 - p) < (1 - \alpha)/4 \) by setting any price below \( 1/2 - \sqrt{\alpha}/2 \). Therefore, the firm is indifferent among all profit maximizing prices \( p \in [1/2 - \sqrt{\alpha}/2, 1/2] \) and both firms using \( \sigma \) constitutes indeed an equilibrium in which both firms expect to earn positive profits.

### 2.2 Strategies

Turning to firms' strategies, let \( \Delta \) denote the set of all probability distributions on the interval \( P \equiv [0, p^*] \), i.e., the set of all right-continuous increasing functions from \( P \) to the unit interval that take the value 1 at \( p^* \). In a game with population uncertainty a strategy function \( \sigma : T \rightarrow \Delta \) substitutes for what in a standard game is a strategy combination (see Myerson, 1998, p. 377). For such a strategy function \( \sigma \), denote by \( \sigma (p | t) \geq 0 \) the conditional probability that a firm will ask \( p \in P \) or less given that it is of type \( t \).

To abbreviate notation, denote by \( \sigma_t \in \Delta \) the evaluation map defined by \( \sigma_t (p) = \sigma (p | t) \) for all \( p \in P \) and all \( t = 1, \ldots, T \).

Conditional on facing \( n \) competitors an active firm will expect to see \( n_1 \) other firms of type \( t = 1 \), \( n_2 \) competitors of type \( t = 2 \), \ldots, and \( n_T \) other firms of type \( t = T \), with probability \( n! \prod_{t=1}^{T} \rho_n (t)^{n_t} / \left( \prod_{t=1}^{T} n_t! \right) \), because of the independence assumption on the type assignment (where \( n = \sum_{t=1}^{T} n_t \)). The probability that a particular price \( p \in P \) is smaller than the price set by a firm of type \( t \) according to a strategy function \( \sigma \in \Delta^T \) is \( 1 - \sigma_t (p) \). For notational brevity define, for each vector \( x \in [0, 1]^T \) and each \( n = 1, 2, \ldots \), the vector \( f^n (x) = (f_1^n (x_1), \ldots, f_T^n (x_T)) \in [0, 1]^T \) by \( f_t^n (x_t) = \rho_n (t) (1 - x_t) \) for all \( t = 1, \ldots, T \).

Then, given a strategy function \( \sigma \), the conditional probability that a particular price \( p \in P \) does not exceed the prices asked by other active firms, given that \( n \) other firms have been drawn, is given by the multinomial expression
\[ F^n_T(\sigma(p)) = \sum_{n_{T-1}=0}^{n-n_T} \sum_{n_T=0}^{n-n_T} \ldots \sum_{n_2=0}^{n-n_T} \sum_{n_1=0}^{n-n_T} \frac{n! f^n_1(\sigma_1(p))^{n-n_T} \prod_{t=2}^{T} f^n_t(\sigma_t(p))}{(n-n_T)! \prod_{t=2}^{T} n_t!} \]

where \( \sigma(p) = (\sigma_1(p), \ldots, \sigma_T(p)) \in [0, 1]^T \). By the binomial law, \( F^n_1(x) = f^n_1(x_1)^n \), and induction yields, for all \( n = 0, 1, \ldots \) and each \( x \in [0, 1]^T \),

\[ F^n_T(x) = \sum_{k=0}^{n} \binom{n}{k} f^n_k(x_1) f^{n-k}_{T-1} = \left[ f^n_k(x_1) + f^n_{T-1}(x) \right]^n = \left[ \sum_{\tau=1}^{T} f^n_{\tau}(x_\tau) \right]^n \] (3)

for all \( t = 2, \ldots, T \). Hence, \( F^n_T(x) = \left[ \sum_{\tau=1}^{T} f^n_{\tau}(x) \right]^n \). The family of functions \( (F^n_T)_{t=1, \ldots, T} \) has the useful property that

\[ F^n_\tau(x) = F^n_{\tau-1}(0) \quad \text{for all } \tau = t, \ldots, T \text{ if and only if} \]

\[ x_\tau = 0 \quad \text{for all } \tau = 1, \ldots, t-1 \quad \text{and} \quad x_\tau = 1 \quad \text{for all } \tau = t, \ldots, T \] (4)

which follows directly from (3).

Since \( F^n_T(\sigma(p)) \) is the probability that \( p \) is the best price among all competitors, conditional on \( n \) active firms (on top of the one under scrutiny), the expected payoff of an active firm of type \( t \) can now be defined: An active firm of type \( t \) obtains \( V_t(p) \) whenever it manages to ask the lowest price among all competitors or when it is the only firm on the market. If at least one active competitor asks a lower price, the firm obtains zero. If it ties with an active competitor at the lowest price in the market, it obtains some fraction of \( V_t(p) \). Assuming for the moment the absence of such ties, the expected profit of a firm of type \( t = 1, \ldots, T \) is then given by

\[ U_t(p, \sigma) = V_t(p) \left[ \sum_{n=0}^{\infty} \pi(n) F^n_T(\sigma(p)) \right] \] (5)

It follows that in equilibrium no active firm can be forced below \( \pi(0) V_t(\overline{p}_t) \geq 0 \), where it bets on being the only supplier and asks the monopoly price.

### 3 Results

The latter observation, that in any equilibrium a firm of type \( t \) must at least obtain \( \pi(0) V_t(\overline{p}_t) \), is the key to the following results. In particular, it implies
that all equilibria must be mixed. For, if the equilibrium strategy function assigns pure choices to all types, then an active firm can discontinuously improve its payoff by slightly undercutting rivals of its own type, provided the strategy function yields positive payoff for its own type. And the latter follows from \( \pi(0)V_t(p_t) > 0 \) whenever \( \pi(0) > 0 \). (Of course, if \( \pi(0) = 1 \), the firm knows that it is a monopolist.)

**Proposition 1** If \( 0 < \pi(0) < 1 \), there exists no equilibrium in pure strategies.

**Proof.** Suppose to the contrary that each firm of type \( t = 1, \ldots, T \) asks \( p_t \in P \) with certainty in equilibrium. Then \( \pi(0) > 0 \) implies that equilibrium payoffs for all types cannot fall below \( \pi(0)V_t(p_t) > 0 \), so that \( V_t(p_t) \geq \pi(0)V_t(p_t) > 0 \) for all \( t = 1, \ldots, T \). With \( \sigma^* \in \Delta^T \) denoting the strategy function that satisfies \( \sigma^*_t(p) = 0 \) for all \( p < p_t \) and \( \sigma^*_t(p) = 1 \) for all \( p \geq p_t \) (an atom at \( p_t \)) for all \( t = 1, \ldots, T \) it follows that

\[
\alpha V_t(p_t) \sum_{n=0}^{\infty} \pi(n) F^n_T(\sigma^*(p_t)) \geq V_t(p) \sum_{n=0}^{\infty} \pi(n) F^n_T(\sigma^*(p))
\]

for some \( \alpha \in [0, 1] \) (for rationing at ties), all \( p \in P \), and all \( t = 1, \ldots, T \). Since by slightly undercutting \( p_T \) a firm of type \( t = T \) can obtain approximately the payoff \( \lim_{\epsilon \searrow 0} U_T(p_T - \epsilon, \sigma^*) \) (avoiding ties with other types), this implies from the equilibrium condition above, (3), \( f_Tn(\sigma^*_T(p_T)) = 0 \), and \( f_Tn(\sigma^*_T(p_T - \epsilon)) = \rho_n(T) \) for \( \epsilon > 0 \) and all \( n = 0, 1, \ldots \), that

\[
\lim_{\epsilon \searrow 0} U_T(p_T - \epsilon, \sigma^*) = V_T(p_T) \sum_{n=0}^{\infty} \pi(n) \left[ \rho_n(T) + \sum_{\tau=1}^{T-1} f_{\tau n}(\sigma^*_\tau(p_T)) \right]^n \leq \alpha V_T(p_T) \sum_{n=0}^{\infty} \pi(n) F^n_{T-1}(\sigma^*(p_T))
\]

This implies from \( V_t(p_t) > 0 \) that

\[
\beta \pi(0) + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \binom{n}{k} \rho_n(T)^k F^n_{T-1}^{n-k}(\sigma^*(p_T)) \leq 0
\]

where \( \beta = 1 - \alpha \), in contradiction to \( \alpha \in [0, 1] \), \( \pi(n) \geq 0 \) for all \( n = 0, 1, \ldots \), \( F^n_{T-1}(\sigma^*(p_T)) \geq 0 \), and

\[
\sum_{n=1}^{\infty} \pi(n) \sum_{k=1}^{n} \binom{n}{k} \rho_n(T)^k F^n_{T-1}(\sigma^*(p_T)) \geq \sum_{n=1}^{\infty} \pi(n) \rho_n(T)^n > 0
\]
because $F^0_t (\sigma (p)) = 1$ for all $(p, \sigma) \in P \times \Delta^T$ and all $t = 1, ..., T$, and $\pi (0) < 1$. \[\blacksquare\]

As a corollary, even in the symmetric case with $T = 1$, there exists no symmetric pure strategy equilibrium as long as $0 < \pi (0) < 1$, even if firms have identical constant marginal costs. For the case without population uncertainty, where firms have constant, but different marginal costs, it is known that mixed equilibria may obtain (Blume, 2003), as with strictly convex cost functions (Hoernig, 2002). With endogenous tie-breaking and different constant marginal costs, pure equilibria exist without population uncertainty (Simon and Zame, 1990), but not with. In general, a pure Bertrand equilibrium does not exists under population uncertainty, even if it exists without.

This raises an existence issue for Nash equilibria in discontinuous games with population uncertainty. Standard existence proofs for discontinuous games (e.g. Dasgupta and Maskin, 1986a,b) do not apply when the payoff function, as in (5), is not linear in mixed strategies. No attempt is made here to supply a general existence proof for equilibria in discontinuous games with population uncertainty. Instead, a mixed equilibrium is constructed for the game at hand.

**Theorem 1** There exists a Nash equilibrium in mixed strategies.

**Proof.** The proof employs a recursive construction, where firms of type $t$ mix over compact intervals $[p^0_t, p^1_t]$. Set $p^1_t = \bar{p}_1$ and $p^0_t = \min \{ \bar{p}_t, p^0_{t-1} \}$ for all $t = 2, ..., T$, where

$$
p^0_t = \min \left\{ p \in P \left| V_t (p) \sum_{n=0}^{\infty} \pi (n) F^n_t (0) \geq V_t (p^1_t) \sum_{n=0}^{\infty} \pi (n) F^n_{t-1} (0) \right. \right\}
$$

for all $t = 1, ..., T$ (note that $F^0_0 (0) = 1$ and $F^n_0 (0) = 0$ for all $n = 1, 2, ...$), and determine $\sigma_t (p) = x$ as the solution to the equations

$$
V_t (p) \sum_{n=0}^{\infty} \pi (n) \left[ f^n_t (x) + \sum_{\tau=1}^{t-1} \rho_n (\tau) \right]^n = V_t (p^1_t) \sum_{n=0}^{\infty} \pi (n) \left[ \sum_{\tau=1}^{t-1} \rho_n (\tau) \right]^n
$$

(6)

for each $p \in [p^0_t, p^1_t]$. For all $p < p^0_t$ set $\sigma_t (p) = 0$ and for all $p > p^1_t$ set $\sigma_t (p) = 1$. (The sum $\sum_{\tau=1}^{t-1} \rho_n (\tau)$ is zero by convention, so for $t = 1$ the right hand side of (6) becomes $V_1 (\bar{p}_1) \pi (0)$.)

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For fixed \( p \in [p^0, p^1] \) the left hand side of (6) is continuous and strictly decreasing in \( x \). The monopoly profit \( V_t(p) \) is strictly increasing on the interval \([p^0, p^1]\), because \( p^1 \leq \overline{p} \). Since \( \left[ \sum_{\tau=1}^{n} \rho_n(\tau) \right]^n = F^n_t (0) > F^n_{t-1} (0) \) for all \( n = 0, 1, ... \) by (3), the left hand side of (6) is at least as large as the right hand side at \( x = 0 \), and, since \( \left[ \sum_{\tau=1}^{n-1} \rho_n(\tau) \right]^n = F^n_{t-1} (0) \) for all \( n = 1, 2, ... \), the left hand side of (6) cannot exceed the right hand side at \( x = 1 \), for all \( p \in [p^0, p^1] \). Therefore, by continuity a unique solution to (6) in the unit interval exists for all \( p \in [p^0, p^1] \) by the intermediate value theorem.

Implicitly differentiating (6) yields

\[
\frac{dx}{dp} = \frac{V_t'(p) \sum_{n=0}^{\infty} \pi(n) \left[ f^n_t(x) + \sum_{\tau=1}^{n-1} \rho_n(\tau) \right]^n}{V_t(p) \sum_{n=0}^{\infty} (n + 1) \pi(n+1) \rho_n(t) \left[ f^n_t(x) + \sum_{\tau=1}^{n-1} \rho_n(\tau) \right]^n} > 0
\]

Moreover, if \( x = 0 \) then, by \( \sum_{k=0}^{n} \binom{n}{k} f^n_t(0)^k f^{n-k}_{t-1}(0) = F^n_t (0) \) for all \( n = 0, 1, ..., (3) \) implies \( p = p^0 \), and if \( x = 1 \) then, by \( \sum_{k=0}^{n} \binom{n}{k} f^n_t(1)^k f^{n-k}_{t-1}(0) = F^n_{t-1} (0) \) for all \( n = 0, 1, ..., (3) \) implies \( p = p^1 \). Hence, \( \sigma_t(p) \) as constructed from (6) is a valid continuous distribution function (without atoms) on the interval \([p^0, p^1]\).

By construction the intervals \([p^0, p^1]\) are ordered by \( p_T^0 \leq p_T^1 \leq p_{T-1}^1 \leq ... \leq p_1^0 \leq p_1^1 = \overline{p} \). Therefore, under the strategy function \( \sigma = (\sigma_t)_{t=1,..,T} \in \Delta^T \) thus constructed, \( \sigma_t(p) = 1 \) for all \( t = 0, 1, ..., T \) and \( \sigma_r(p) = 0 \) for all \( t = 1, ..., t-1 \), for any \( p \in [p^0, p^1] \). It follows from (4) that \( F_T^n(\sigma(p)) = \sum_{k=0}^{n} \binom{n}{k} f^n_t(\sigma_t(p))^k F^{n-k}_{t-1}(0) \) for all \( p \in [p^0, p^1] \). Therefore, for all \( p \in [p^0, p^1] \), the equilibrium payoff

\[
U^*_t(\sigma) = V_t(p) \sum_{n=0}^{\infty} \pi(n) F^n_t(\sigma(p)) = V_t(p_1^1) \sum_{n=0}^{\infty} \pi(n) \left[ \sum_{\tau=1}^{t-1} \rho_n(\tau) \right]^n
\]

is constant. For any \( \tau \neq t \) the payoff on the interval \([p^0, p^1]\) satisfies (from (5))

\[
U_t(p, \sigma) = \frac{V_t(p) \sum_{n=0}^{\infty} \pi(n) \left[ \sum_{\tau=1}^{t-1} \rho_n(\tau) \right]^n}{V_t(p_1^1) \sum_{n=0}^{\infty} \pi(n) \left[ \sum_{\tau=1}^{t-1} \rho_n(\tau) \right]^n}
\]

Because \( \partial (V_t(p) / V_t(p)) / \partial p = [V_t V'_t - V'_t V_t] / V_t^2 \), \( V_t(p) \) strictly increases with \( \tau \), and \( V'_t(p) \) strictly decreases with \( \tau \), the payoff \( U_t(p, \sigma) \) is increasing in \( p \in [p^0, p^1] \) whenever \( \tau = 1, ..., t-1 \) and it is decreasing in \( p \in [p^0, p^1] \) whenever \( \tau = t + 1, ..., T \). Finally, for any \( p \in (p_1^1, p_{t-1}^0) \) the term \( \sum_{n=0}^{\infty} \pi(n) F^n_t(\sigma(p)) \) is constant, implying that the payoff \( U_t(p, \sigma) \) decreases
in \( p \) for all \( \tau = t, \ldots, T \) (as \( \bar{p}_\tau \leq p^1_t \) for \( \tau \geq t \)), and it increases in \( p \) for all \( \tau = 1, \ldots, t - 1 \) (as \( p^0_{t-1} \leq \bar{p}_\tau \) for \( \tau < t \)), for all \( t = 1, \ldots, T \). These properties verify that the pure best replies of type \( t \) are precisely the prices \( p \in [p^0_t, p^1_t] \). Thus, an equilibrium has been constructed. \( \blacksquare \)

Unlike Proposition 1 the theorem does not take the hypothesis that \( 0 < \pi(0) < 1 \). This is because mixed strategy equilibria may occasionally be pure. For instance, if \( \pi(0) = 1 \), a simple computation shows that the probability distributions \( \sigma_t \) constructed in the proof of Theorem 1 degenerate to atoms with unit mass at the monopoly prices \( \bar{p}_t \) for all types \( t = 1, \ldots, T \). That is, implicitly the theorem covers cases, where pure strategy equilibria exist, like when the firm knows that it is a monopolist.

Two more points are worth noting. First, as (6) and (7) reveal, what counts for the equilibrium payoff of a firm of type \( t \) is them as so fles se ffi e c i e n t types conditional upon the number \( n \) of competitors, \( \sum_{\tau=1}^{n-1} \rho_n(\tau) \). If there were a continuum of types, this sum would be replaced by an integral over less efficient types, but otherwise little would change in the proof. In this sense the assumption of a finite number of types constitutes no serious loss of generality.

Second, Theorem 1 makes no claim about uniqueness. In fact, there are special cases for which other equilibria are known to exist. For instance, if there is no population uncertainty, i.e. \( \pi(m) = 1 \) for some \( m > 1 \) and \( \pi(n) = 0 \) for all \( n \neq m \), the present model is covered by Spulber’s (1995) model (where the type distribution in the latter is a step function). Yet, Spulber identifies a pure strategy equilibrium. \(^5\)

Since the proof of Theorem 1 constructs an equilibrium, its properties can be studied. A first comparative static exercise is obvious from the proof of Theorem 1: Since \( p^1_{t+1} \leq p^0_t \) for \( t = 1, \ldots, T - 1 \), the equilibrium price distribution of more efficient firm types is supported at lower prices than the one of less efficient types. Therefore, in this equilibrium more efficient types will tend to ask lower prices and sell higher quantities (since \( Q \) is decreasing in the price) in the event that they win the market.

Furthermore, Proposition 1 already shows that an outside observer will see price dispersion on a market with population uncertainty, because equilibria are in mixed strategies. The proof of Theorem 1 qualifies the stochastic nature of prices. Firms randomize over bounded supports (compact inter-

\(^5\) In fact, Theorem 1 qualifies Spulber’s uniqueness claim (1995, Proposition 2, p. 5) as only applying to pure strategy equilibria.
vals), and different firm types employ different supports. Therefore, observing a price offer does contain information about the firm’s efficiency type. That is, even though prices are “noisy,” they do contain information.

The next result adds further comparative statics. Its first part says that, unlike in the classical Bertrand case, equilibrium profits are positive, whenever there is a chance to be a monopolist. Its second part says, as expected, that more efficient firms do better than less efficient ones. Its third part is a comparative static exercise that concerns the expected number of competitors. To make precise what it means to expect more rivals, say that a probability distribution $\hat{\pi}$ on the nonnegative integers (first-order) stochastically dominates another, $\pi$, if

$$\sum_{n=0}^{m} \hat{\pi}(n) \leq \sum_{n=0}^{m} \pi(n) \text{ for all } m = 0, 1, ... \quad (8)$$

If this is the case, the mean $\sum_{n=0}^{\infty} n \hat{\pi}(n)$ with respect to $\hat{\pi}$ is at least as large as the mean $\sum_{n=0}^{\infty} n \pi(n)$ with respect to $\pi$. Stochastic dominance is thus the appropriate notion of a “larger expected number of competitors.”

**Proposition 2** In the equilibrium constructed in Theorem 1 profits

(a) are positive for all active firms, whenever $\pi(0) > 0$;

(b) increase with the types of active firms;

(c) decline when the expected number of competitors grows.

**Proof.** (a) First, whenever $\pi(0) > 0$, equilibrium profits for all firms are positive, i.e. $U_t^*(\sigma) \geq \pi(0) V_t(\bar{p}_t) > 0$ for all $t = 1, ..., T$. To see this, observe that if $\bar{p}_t \leq p_{t-1}^0$ then $p_t^1 = \bar{p}_t$ implies

$$U_t^*(\sigma) = V_t(\bar{p}_t) \sum_{n=0}^{\infty} \pi(n) F_{t-1}^n(0) \geq V_t(p) \sum_{n=0}^{\infty} \pi(n) F_{t-1}^n(0)$$

for all $p \in P$, implying that $U_t^*(\sigma) \geq V_t(p) \sum_{n=0}^{\infty} \pi(n) F_{t-1}^n(0) \geq \pi(0) V_t(p)$ for all $p \in P$ and all $t = 0, ..., t-1$, in particular, $U_t^*(\sigma) \geq \pi(0) V_t(\bar{p}_t) > 0$. If $p_t^1 = p_{t-1}^0 < \bar{p}_t$, then on the interval $[p_{t-1}^0, p_t^1]$ the expected profit $U_t$ can be written as

$$U_t(p, \sigma) = V_t(p) \sum_{n=0}^{\infty} \pi(n) \left[ \sum_{\tau=1}^{t-2} \rho_n(\tau) + \rho_n(t-1)(1 - \sigma_{t-1}(p)) \right] =$$

$$V_t(p) \frac{V_{t-1}(p_{t-1})}{V_{t-1}(p)} \sum_{n=0}^{\infty} \pi(n) F_{t-2}^n(0) = \frac{V_t(p)}{V_{t-1}(p)} U_{t-1}^*(\sigma)$$
which is a decreasing function of \( p \) (because \( \partial (V_t (p) / V_{t-1} (p)) / \partial p < 0 \)) and, therefore, attains its maximum at \( \hat{p}_{t-1}^0 \). Hence, \( U_t^* (\sigma) = U_t (\hat{p}_{t-1}^0, \sigma) \geq U_t (p, \sigma) \geq V_t (p) \sum_{n=0}^{\infty} \pi (n) F_{n-2}^0 (0) \), since \( V_{t-1} (p_{t-1}^1) \geq V_{t-1} (p) \) for all \( p \in [p_{t-1}^0, p_{t-1}^1] \). Since, by the same logic, for any \( p_{t}^0 > p_{t-1}^0 \) the expected profit satisfies

\[
U_t (p_{t}^0, \sigma) = V_t (p_{t}^0) \sum_{n=0}^{\infty} \pi (n) F_{n}^0 (0) \geq V_t (p) \sum_{n=0}^{\infty} \pi (n) F_{n-2}^0 (0)
\]

for all \( \tau = 1, \ldots, t-1 \) and all \( p \in [p_{t}^0, p_{t}^1] \) (again because \( \partial (V_t (p) / V_{t} (p)) / \partial p < 0 \) for all \( \tau < t \)), the inequality \( U_t^* (\sigma) \geq V_t (p) \sum_{n=0}^{\infty} \pi (n) F_{n-2}^0 (0) \) holds for all \( p \geq p_{t-1}^0 \), in particular, \( U_t^* (\sigma) \geq V_t (\bar{p}_t) \sum_{n=0}^{\infty} \pi (n) F_{n-2} (0) \geq \pi (0) V_t (\bar{p}_t) > 0 \).

(b) Second, equilibrium profits rise with the type. To see this, observe that by (7), for all \( t = 2, \ldots, T \),

\[
U_t^* (\sigma) = \frac{V_t (p_{t}^1)}{V_{t-1} (p_{t-1}^1)} V_{t-1} (p_{t-1}^1) \sum_{n=0}^{\infty} \pi (n) \left[ \sum_{\tau=1}^{t-2} \rho_n (\tau) \right] \geq \frac{V_t (p_{t-1}^0)}{V_{t-1} (p_{t-1}^0)} V_{t-1} (p_{t-1}^0) \sum_{n=0}^{\infty} \pi (n) \left[ \sum_{\tau=1}^{t-2} \rho_n (\tau) \right] > \frac{V_{t-1} (p_{t-1}^1)}{\pi (n)} \sum_{n=0}^{\infty} \pi (n) \left[ \sum_{\tau=1}^{t-2} \rho_n (\tau) \right] = U_{t-1}^* (\sigma)
\]

because either \( p_{t}^1 = p_{t-1}^0 \) or \( p_{t}^1 \neq p_{t-1}^0 \) implies \( p_{t}^1 = \bar{p}_t \), so that \( V_t (p_{t}^1) = V_t (\bar{p}_t) \geq V_t (p_{t-1}^0) \), verifying the first inequality, and \( V_t (p) > V_{t-1} (p) \) for all \( p \in [p_{t-1}^0, p_{t}^1] \), verifying the second inequality.

(c) Third, equilibrium profits fall as the expected number of competitors rises. To see this, first note that, whenever \( \hat{\pi} \) stochastically dominates \( \pi \), then \( \sum_{n=0}^{\infty} g (n) \hat{\pi} (n) \leq \sum_{n=0}^{\infty} g (n) \pi (n) \) holds, for any real-valued decreasing function \( g \) on the nonnegative integers. As a consequence, if \( \hat{\pi} \) stochastically
dominates \( \pi \), then, for \( t = 1 \),

\[
U_t^* (\sigma) = V_t (p_1^0) \sum_{n=0}^{\infty} \pi (n) \rho_n (1)^n = \pi (0) V_t (\overline{p}_t) \geq V_t (\overline{p}_t^{(t)}) \sum_{n=0}^{\infty} \hat{\pi} (n) \rho_n (1)^n = \hat{U}_1^* (\hat{\sigma})
\]

denoting variables under \( \hat{\pi} \) by a hat. Applying induction, assume now that \( \hat{U}_{t-1}^* (\hat{\sigma}) \leq U_{t-1}^* (\sigma) \).

For types \( t = 2, ..., T \) the following cases have to be distinguished. If \( \overline{p}_t \leq \min \{ \overline{p}_0^{(t)}, \overline{p}_1^{(t)} \} \), then \( \hat{p}_t = p_t = \overline{p}_t \) and \( \hat{U}_t^* (\hat{\sigma}) = V_t (\overline{p}_t) \sum_{n=0}^{\infty} \hat{\pi} (n) F_{t-1}^n (0) \leq V_t (\overline{p}_t) \sum_{n=0}^{\infty} \pi (n) F_{t-1}^n (0) = U_t^* (\sigma) \) by (8) yields the desired conclusion. If \( \overline{p}_0 < \overline{p}_t \leq \overline{p}_1^{(t)} \), then \( \hat{p}_t = \overline{p}_0 < p_t = \overline{p}_t \) implies

\[
\hat{U}_t^* (\hat{\sigma}) = V_t (\overline{p}_1^{(t-1)}) \sum_{n=0}^{\infty} \hat{\pi} (n) F_{t-1}^n (0) \leq V_t (\overline{p}_1^{(t-1)}) \sum_{n=0}^{\infty} \pi (n) F_{t-1}^n (0) = U_t^* (\sigma)
\]

by (8), as desired. If \( \overline{p}_0 < \overline{p}_1^{(t-1)} \), then

\[
\hat{U}_t^* (\hat{\sigma}) = V_t (\overline{p}_1^{0}) \sum_{n=0}^{\infty} \hat{\pi} (n) F_{t-1}^n (0) <
\]

\[
V_t (\overline{p}_1^{0}) \sum_{n=0}^{\infty} \pi (n) F_{t-1}^n (0) \leq V_t (\overline{p}_1^{0}) \sum_{n=0}^{\infty} \pi (n) F_{t-1}^n (0) = U_t^* (\sigma)
\]

follows from \( V_t (\overline{p}_1^{0}) < V_t (\overline{p}_1^{0}) \) and (8).

The remaining two possibilities are more involved. If \( p_{t-1}^{0} \leq \overline{p}_1^{0} \leq \overline{p}_t \), then for all \( p \in [\overline{p}_1^{0}, \overline{p}_t] \)

\[
\hat{U}_t (p, \sigma) = V_t (p) \sum_{n=0}^{\infty} \hat{\pi} (n) \left[ \sum_{\tau=1}^{t-2} \rho_n (\tau) + \rho_n (t-1) (1 - \sigma_{t-1} (p)) \right] =
\]

\[
V_t (p) \frac{\hat{U}_{t-1}^* (\hat{\sigma})}{V_{t-1} (p)} \leq \frac{V_t (p)}{V_{t-1} (p)} U_{t-1}^* (\sigma) = U_t (p, \sigma)
\]

using \( \hat{U}_{t-1}^* (\hat{\sigma}) \leq U_{t-1}^* (\sigma) \). In particular, \( \hat{U}_t (\overline{p}_1^{0}, \sigma) = \hat{U}_t^* (\hat{\sigma}) \leq U_t (\overline{p}_1^{0}, \sigma) \), and the latter satisfies \( U_t (\overline{p}_1^{0}, \sigma) \leq U_t (\overline{p}_0^{t}, \sigma) = U_t^* (\sigma) \), because \( U_t (p, \sigma) \)
is decreasing on the interval \([p_{t-1}^0, p_{t-1}^1]\) and \(p_{t-1}^0 \leq \tilde{p}_{t-1}^0\) by hypothesis. Finally, if \(p_{t-1}^0 \leq \bar{p}_t < \tilde{p}_{t-1}^0\), then \(\tilde{p}_t = \bar{p}_t\) and
\[
\sum_{n=0}^{\infty} \hat{\pi} (n) F_{t-1}^n (0) \leq \frac{V_{t-1} (p_{t-1}^0)}{V_{t-1} (p_{t-1}^0)} \sum_{n=0}^{\infty} \pi (n) \left[ \sum_{\tau=1}^{t-1} \rho_n (\tau) \right] < \sum_{n=0}^{\infty} \pi (n) F_{t-1}^n (0)
\]
(from \(p_{t-1}^0 < \tilde{p}_{t-1}^0\)) implies
\[
\hat{U}_t^* (\hat{\sigma}) = V_t (\bar{\pi}_t) \sum_{n=0}^{\infty} \hat{\pi} (n) F_{t-1}^n (0) \leq V_t (\bar{\pi}_t) \frac{V_{t-1} (p_{t-1}^0)}{V_{t-1} (p_{t-1}^0)} \sum_{n=0}^{\infty} \pi (n) F_{t-1}^n (0) < \frac{V_t (\bar{\pi}_t)}{V_{t-1} (\bar{\pi}_t)} U_{t-1}^* (\sigma) \leq \frac{V_t (p_{t-1}^0)}{V_{t-1} (p_{t-1}^0)} U_{t-1}^* (\sigma) = U_t^* (\sigma)
\]
because \(V_t (p) / V_{t-1} (p)\) is decreasing on the interval \([p_{t-1}^0, p_{t-1}^1]\).

Thus, if \(\hat{\pi}\) stochastically dominates \(\pi\), then \(\hat{U}_t^* (\hat{\sigma}) \leq U_t^* (\sigma)\) for all \(t = 1, \ldots, T\), i.e., a higher number of expected rivals drives equilibrium profits down.

Even though equilibrium profits decline as the firm expects more competitors, according to Proposition 2(c), there is in general no guarantee that they are driven to zero as the expected number of rivals diverges. Indeed, they will not, as long as \(\pi (0) > 0\), according to Proposition 2(a). That growing competition wipes out profits, therefore, necessarily requires that the chances of being a monopolist, \(\pi (0)\), go to zero. Since \(\pi\) is a primitive of the model, little can be said about this without additional structure. Yet, as will be shown below, in some prominent instances this is indeed the case.

4 Special Cases

In this section a few interesting special cases of the equilibrium constructed in Theorem 1 are presented. Those are cases that have been studied in the literature to a certain extent. The main advantage of these cases is that the equilibrium strategy can be computed in closed form, thus allowing for more comparative static insights.

4.1 No Population Uncertainty

First, consider the case of no population uncertainty, where \(\pi (m) = 1\) for some \(m > 0\) and \(\pi (n) = 0\) for all \(n \neq m\). Then the payoff from (5) becomes
\[ U_t (p, \sigma) = V_t (p) \left[ \sum_{\tau=1}^{T} f_n^m (\sigma_\tau (p)) \right]^m \] and (6) yields

\[ \sigma_t (p) = 1 - \frac{\sum_{\tau=1}^{t-1} \rho_m (\tau)}{\rho_m (t)} \left[ \left( \frac{V_t (p_1^t)}{V_t (p)} \right)^{1/m} - 1 \right] \]

with \( p_0^t = \min \left\{ p \in P \left| V_t (p) \left[ \sum_{\tau=1}^{T} \rho_m (\tau) \right]^m \geq V_t (p_1^t) \left[ \sum_{\tau=1}^{t-1} \rho_m (\tau) \right]^m \right. \} \) and \( p_1^t = \min \{ \bar{\sigma}_t, p_{t-1}^0 \} \) for all \( t = 2, \ldots, T \), \( p_0^1 = \min \{ p \in P | V_1 (p) \geq 0 \} \), and \( p_1^1 = \bar{\sigma}_1 \). Hence, the least efficient firms (of type \( t = 1 \)) make no profits and play a pure strategy with \( \sigma_1 (p) = 0 \) for all \( p < p_0^1 \) and \( \sigma_1 (p) = 1 \) for all \( p \geq p_0^1 \) (an atom at \( p_0^1 \)). Firms with a more efficient type \( t > 1 \) make positive profits, but less so the higher \( m \) is. As \( m \) goes to infinity, profits are wiped out for all types. Thus, with uncertainty about the competitors’ cost functions, but no population uncertainty, equilibrium profits are positive for finitely many firms, but go to zero as the number of firms becomes arbitrarily large.

In fact, more can be said about the case of large \( m \). Since the term \( (V_t (p_1^t)) / V_t (p) \)^{1/m} converges to 1 as \( m \) goes to infinity, the equilibrium price distributions for all types \( t \) converge to an atom of unit mass (at \( p_0^t \)). By the previous observation, this implies that equilibrium prices approach average costs as the number \( m \) of firms becomes arbitrarily large, for each type \( t = 1, \ldots, T \). Hence, in a large market price dispersion will become small and expected prices will approximate average costs. This conclusion is in line with the pure strategy equilibrium identified by Spulber (1995, Proposition 4).

With \( T = 2 = m+1 \) and constant, but different marginal costs this version of the model yields a stochastic variant of the mixed Bertrand equilibrium described in Blume (2003). Because of uncertainty about cost functions of rivals, however, it is here the more efficient firms that mix, while inefficient firms play a pure strategy. (In Blume, 2003, the situation is the other way around.) Yet, the equilibrium constructed in the proof of Proposition 1 remains valid even when cost functions are strictly convex. This gives a mixed Bertrand equilibrium for the case of strictly decreasing returns to scale and uncertainty about the competitors’ cost functions.
4.2 No Cost Uncertainty

Second, consider the symmetric case of a single type, $T = 1$. Then, according to (6), all firm mix according to

$$V(p) \sum_{n=0}^{\infty} \pi(n) [1 - \sigma(p)]^n = \pi(0) V(\overline{p})$$

on the interval $[p^0, \overline{p}]$, where $p^0 = \min \{ p \in P | V(p) \geq \pi(0) V(\overline{p}) \}$ (suppressing the subscript for the single type). Since the right hand side of (6) equals the right hand side of (7) and $V(\overline{p}) > 0$, it follows that equilibrium profits will go to zero if and only if the probability of being a monopolist, $\pi(0)$, does. This is a condition that in a more encompassing model would be determined by the decision of firms whether or not to enter the market. Here, the distribution $\pi$ is a primitive, so that little can be said about it in general. Still, for some prominent cases from the literature the condition holds true.

With $\pi(n) = \binom{m-1}{n} (1 - \alpha)^{m-1-n} \alpha^n$ for all $n = 0, ..., m - 1$ and $\pi(n) = 0$ for all $n = m, m + 1, ...$ for some $\alpha \in [0, 1]$ and with $C(q) = 0$ for all $q \geq 0$ the model with a single type is exactly the model of Janssen and Rasmusen (2002). In their case $1 - \alpha$ is the probability (uniform and independent across firms) that a firm is hit by a shock that prevents it from participating in the market. Accordingly, (6) reads

$$V(p) [1 - \alpha \sigma(p)]^{m-1} = (1 - \alpha)^{m-1} V(\overline{p})$$

$$\Rightarrow \sigma(p) = \frac{1}{\alpha} - \frac{(1 - \alpha)}{\alpha} \left( \frac{V(\overline{p})}{V(p)} \right)^{\frac{1}{m-1}}$$

Again, equilibrium profits are positive, but fall as $m$ rises, and approach zero as $m$ goes to infinity. Moreover, $\lim_{m \to \infty} (V(\overline{p}) / V(p))^{1/(m-1)} = 1$ implies that the equilibrium price distribution converges to an atom of unit mass, i.e., again price dispersion becomes small as the number of competitors grows. Since also equilibrium profits approach zero as $m \to \infty$, expected equilibrium prices approximate marginal (equal to average) costs in a large market.\(^6\)

Finally, with $m = 1$, i.e. no population uncertainty, this version yields the classical Bertrand case, where all firms set a price equal to marginal cost and profits are zero.

\(^6\) It is not difficult to obtain a similar conclusion if, for instance, the distribution of the number of competitors is geometric, i.e. $\pi(n) = (1 - \delta) \delta^n$ for some $\delta \in (0, 1)$, rather than binomial as in the model by Janssen and Rasmusen (2002).
4.3 Poisson Game

Finally, an interesting special case emerges if $\pi$ is assumed to be Poisson, 
$\pi(n) = e^{-\mu} \frac{n^m}{n!}$ for all $n = 0, 1, \ldots$ (a “Poisson game” in the sense of Myerson, 1998, 2000) and the type distribution does not depend on $n$, i.e. $\rho_n = \rho$ for all $n = 1, 2, \ldots$ Then,

$$\sum_{n=0}^{\infty} \pi(n) \left[ \sum_{\tau=1}^{t-1} \rho(\tau) \right]^n = e^{-\mu} \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \mu \sum_{\tau=1}^{t-1} \rho(\tau) \right]^n = \exp \left\{ -\mu \sum_{\tau=t}^{T} \rho(\tau) \right\}$$

and (6) yields $\sigma_t(p) = 1 - \left[ \ln V_t(p_t^1) - \ln V_t(p) \right] / (\mu \rho(t))$ for all $p \in [p_t^0, p_t^1]$, and (7) reads

$$U_t^*(\sigma) = V_t(p_t^1) \exp \left\{ -\mu \sum_{\tau=t}^{T} \rho(\tau) \right\}$$

for all $t = 1, \ldots, T$. Hence, equilibrium profits are clearly decreasing in the mean $\mu$ and approach zero as $\mu$ goes to infinity. Moreover, as $\mu$ becomes large, the distributions $\sigma_t$ become more and more concave—putting more mass at lower prices—and approach an atom of mass 1 at $p = p_t^1$ as $\mu \rightarrow +\infty$. Thus, again in the limit each firm type plays a pure strategy, pricing at average cost.

5 Conclusions

The Bertrand model holds that two firms are enough to drive prices to marginal costs on a market for a homogeneous commodity—provided it is common knowledge how many firms compete and that all have identical constant marginal costs. This is a paradoxical conclusion, because equilibrium profits are identically zero and, therefore, do not provide any entry incentives. Dropping the common knowledge assumption about either the number of competitors or about (identical constant marginal) cost (or both) resolves this paradox: Expected equilibrium profits are positive with more efficient firms making more. Still, and in line with economic intuition, more firms yield more competition and lower equilibrium profits. This holds both with population uncertainty and with uncertainty about cost functions. The present paper combines these two sources of uncertainty in a single model that also allows for a potentially infinite number of firms.
A similar model could also be applied to procurement auctions with a variable number of bidders. The difference is that in reverse auctions the quantity sold is usually not price dependent, i.e., there is no continuous demand function anymore. Occasionally, when such auctions are used to source buyer-designed goods or services, the ultimate quantity is decided ex-post in the light of the winning bid. In that case the present model could be applied directly. But most of the time the desired quantity is fixed ex-ante, implying that $Q$ is a constant. In that case a monopoly price does not exist. With an upper bound on feasible prices the arguments in the present paper could potentially still be adapted to cover that case. This is left for future research, though.

References


