Beyond the Sharpe Ratio: Performance Measurement with an Economic Index of Riskiness

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We propose a performance measure that generalizes the Sharpe ratio. The two measures are equivalent if returns are normally distributed. Moreover, they are asymptotically equivalent as the underlying distributions converge to the normal distribution. The new performance measure is obtained from the recently introduced economic index of riskiness of Aumann and Serrano (2008) [*Journal of Political Economy*, **116**, pp. 810-836] and, in contrast to the Sharpe ratio, respects stochastic dominance. Furthermore, we propose a parametric and a non-parametric estimator for the new performance measure and provide an empirical illustration using mutual fund data.

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1 Introduction

Performance measures are important tools in theoretical and applied economics. They induce a total (or partial) order of investment opportunities such that agents can reduce their decisions regarding these investments to a simple comparison of these coefficients. Such decision for instance are concerned with ranking investment opportunities or the evaluation of money managers, i.e. hedge funds managers.

The *Sharpe ratio* (introduced as and also called *reward-to-variability ratio*), proposed by Sharpe (1966, 1994), is one of the most prominent performance measures. It is the ratio of the mean over the standard deviation of the expected excess return of an investment opportunity. It thus corrects the expected return by taking into account a specific type of risk taken by the investor. It implicitly builds on the assumption that the mean and standard deviation of an investment opportunity completely characterize its distribution, as it would be the case for the normal distribution, or that these properties are the only ones an investor cares about. However, it is well known that financial returns are very often (left) skewed and have excess kurtosis, i.e. more probability mass is in the tails of the distribution than implied by the normal distribution. Empirical and experimental studies have shown that it is unlikely that investors do not care about this (cf. Golec and Tamarkin, 1998; Harvey and Siddique, 2000). Especially in the fast growing and unregulated field of hedge funds the returns show significant non-normal characteristics which may be due to dynamic trading strategies in derivatives, e.g. see Agarwal and Naik (2004), Malkiel and Saha (2005), Goetzmann, Ingersoll, Spiegel, and Welch (2007) and the references therein. Moreover, very often hedge funds also exhibit a larger Sharpe ratio than mutual funds. However, since the Sharpe ratio ignores higher order moments the question arises whether the superior performance of hedge funds (in the sense of the Sharpe ratio) remains if one accounts for non-normality.

This has led to the development of various new performance measures which take into account these stylized facts of financial returns. Most of them either replace the mean with a different reward measure or they substitute the standard deviation with a new measure of the (relevant) risk taken by the investor, or both. However, most of them are proposed in a rather ad-hoc way and their economic implications remain unclear.

In this paper we propose a new performance measure based on the *economic index of riskiness* proposed by Aumann and Serrano (2008) (AS index thereafter). In fact, the performance measure is obtained by dividing the mean of an investment opportunity by its AS index. We will refer to this measure as the *economic performance measure* (EPM). Contrary to the Sharpe ratio and other performance measures, the EPM fulfills the natural requirement, that it be monotone with respect to stochastic dominance. Moreover, if investment returns are normally distributed, the EPM and the Sharpe ratio produce the same ranking of these investments (cf. Aumann and Serrano, 2008). In this sense the EPM generalizes the Sharpe ratio.

Moreover, we extend the continuity result of Aumann and Serrano (2008) and show that, if the distribution of the returns converges to the normal distribution, the EPM converges to two times the squared Sharpe ratio. Thus, the EPM also asymptotically induces the same ranking as the Sharpe ratio. This property is especially appealing in connection with the *aggregational Gaussianity* property of financial returns. This property states that for decreasing sampling frequency, e.g. going from daily to monthly and down to yearly returns, the return distribution approximates the normal distribution, see e.g. Cont (2001) and Rydberg (2000).

We propose a parametric and a non-parametric moment estimator for the EPM. For parametric estimation we assume that returns follow a normal inverse Gaussian (NIG) distribution proposed by Barndorff-Nielsen (1997). As the NIG distribution is analytically very tractable and has several attractive properties it has found widespread use in financial applications. It allows to model skewness and semi-heavy tails. We derive a closed form expression for the EPM of NIG-distributed random variables (e.g. excess returns) in terms of the first four moments. This makes explicit the dependence on skewness and kurtosis and enables us to provide a moment estimator for the EPM that is virtually as easy to compute as that for the Sharpe ratio. The crucial idea for non-parametric estimation is to use a moment condition that corresponds to the defining equation of the AS index. Results on asymptotic normality can readily be inferred from the literature on the method of moments.

We apply our two estimators to rank mutual funds via the EPM and compare results with a Sharpe ratio ranking. While the different estimation techniques for the EPM lead to a very similar ranking, there is a considerable difference to the ranking generated by the Sharpe ratio. If a fund's return distribution has relatively low (high) skewness and/ or relatively high (low) excess kurtosis, the fund is typically ranked lower (higher) by the EPM than by the Sharpe ratio.

The remainder of the paper is structured as follows. In the next section we introduce the economic performance measure, derive properties of this new index, and discuss its relation to other performance measures. Estimators for the EPM are suggested in Section 3. Section 4 provides an empirical illustration using mutual fund returns. Section 5 concludes and supplementary calculations are contained in the appendix.

2 An economic performance measure

Let $\tilde{r}$ denote the (stochastic) return of an investment portfolio, r^f the (deterministic) risk-free rate and r the resulting (stochastic) excess return. The economic performance measure is defined as the expected excess return relative to the AS index of riskiness of this return:

$$\mathsf{EPM}\,(r) = \frac{\mathbb{E}\,(r)}{AS(r)} = \frac{\mathbb{E}\,(\tilde{r}) - r^f}{AS(\tilde{r} - r^f)}. \tag{2.1}$$

In the following Section 2.1 we briefly review the AS index and then discuss the properties of the EPM in Section 2.2. We generalize the continuity property of the AS index shown by Aumann and Serrano (2008) and extend this property to the EPM. In Section 2.3 we discuss how the EPM relates to other performance measures.

2.1 The Aumann-Serrano index of riskiness

Aumann and Serrano's (2008) index of riskiness is an axiomatic approach to assign a meaning to the word *risky*. It enables a decision maker to asses which of two investments is *riskier* without referring to a specific preference order. Similar to the standard deviation or *value at risk* (VaR), the AS index summarizes the properties of a *gamble* in a single number, thereby making comparisons very easy.

Note that although the AS index was proposed for gambles in terms of absolute outcomes, it can straightforwardly be applied to excess returns. Indeed, the excess return $\tilde{r} - r^f$ can be regarded as the outcome of a zero investment strategy consisting in borrowing \$1 and investing it in a risky asset for a given time span. In the remaining part of this article we will identify "the gamble" or the "excess return" with this zero investment strategy.

Let an index Q be a mapping that assigns a positive real number to each excess return/random variable with values in $\mathbb{R}$. Of course, not every index provides a meaningful summary of an investment's "riskiness". Aumann and Serrano (2008) argue that a reasonable risk index should satisfy the following two axioms:

D *Duality:* If i and j are two agents, such that i is uniformly more risk averse than j,[1] and if i accepts an excess return $r^{(A)}$ at wealth w and $Q(r^{(A)}) > Q(r^{(B)})$, then j accepts the excess return $r^{(B)}$ at wealth w.

H *Homogeneity:* For any positive real number t, it holds that $Q(tr^{(A)}) = tQ(r^{(A)})$.

Here, for $t > 0$, $tr^{(A)}$ is the gamble that results from $r^{(A)}$ by multiplying every outcome of $r^{(A)}$ by t. It is quite natural to think that, if the stakes are doubled, the risk is also doubled,

[1] Agent i is called uniformly more risk averse than agent j, if the fact that i accepts an excess return at some wealth implies that j accepts that excess return at any wealth.

i.e. to require homogeneity. Note that axiom **D** is actually a mild requirement on an index, since the numbers assigned to two gambles only have an implication for an agent's decision, if strong preconditions are satisfied. Axiom **D** basically says, that if an agent accepts an excess return $r^{(A)}$, then a less risk averse agent should accept an excess return $r^{(B)}$ that is less risky than $r^{(A)}$ according to the index.

Aumann and Serrano (2008) showed that any two indexes that satisfy **D** and **H** are positive multiples of each other. Moreover, a specific index that satisfies **D** and **H** can be obtained as the positive solution to the equation

$$\mathbb{E}\left(\mathrm{e}^{-\frac{r^{(A)}}{s}}\right) = 1. \tag{2.2}$$

We refer to the solution of (2.2) as the AS index of $r^{(A)}$, $\mathsf{AS}\left(r^{(A)}\right)$.

The AS index of riskiness is objective in the sense that it does not depend on the preferences of an individual agent. Furthermore, as Aumann and Serrano (2008) demonstrate, it is (strictly) monotone with respect to first and second order stochastic dominance, which is an important property.

It remains to clarify the question about the existence of the AS index. If the excess return r takes on only finitely many values Aumann and Serrano (2008) show that the following assumptions are necessary and sufficient for a unique positive solution to (2.2):

(i) possibly negative outcomes, i.e. $\mathbb{P}\left(r < 0\right) > 0$, and

(ii) expected value greater than zero, i.e. $\mathbb{E}\left(r\right) > 0$.

However, for continuous distributions, i.e. distributions with uncountable many outcomes, the existence of this index is more delicate. The *moment generating function* (mgf) of an excess return r is defined by

$$M_r\left(t\right) = \mathbb{E}\left(\mathrm{e}^{tr}\right).$$

Furthermore, let $I_r = \{t \in \mathbb{R} : M_r(t) < \infty\}$ and $l = \inf I_r < 0$. Homm and Pigorsch (2011) have shown that in addition to (i) and (ii) the following condition must also be satisfied to ensure the existence of the AS index:

(iii) $l \notin I_r$ or $M_r(l) \geq 1$.

Note that for excess returns with finitely many outcomes $l = -\infty$ and (iii) is satisfied.

2.2 Properties of the economic performance measure

Having introduced the AS index, we now provide properties of the EPM. Some of these are easily inferred from the properties of the AS index. Moreover, we derive a general form of continuity for the AS index and extend it to the EPM. In contrast to Aumann and Serrano (2008), this *generalized continuity* also applies to excess returns with infinitely many possible outcomes, which is typically the case in financial return modeling.

Scale Invariance

Both the numerator and the denominator of the EPM (2.1) are homogeneous, so that the EPM is scale invariant. Therefore the scale of the investment can be set to \$1 without loss of generality, identifying the excess return as the zero investment using \$1.

Interpretation

The EPM can be interpreted as a measure of *reward to required capital*. Homm and Pigorsch (2011) show that, for $0 < p < 1$ and p close to zero, $\log(1/p)\mathsf{AS}(r)$ is approximately equal to the minimum required initial wealth that assures no bankruptcy with probability $1-p$ when playing the zero investment strategy with excess return r infinitely often.[2] We do not use the factor $\log(1/p)$, however, since it is irrelevant for ranking purposes.

Stochastic dominance

Beyond the fact that the EPM uses a risk measure that is (strictly) monotone with respect to first ("$\overset{①}{\succ}$") and second order ("$\overset{②}{\succ}$") stochastic dominance, the EPM itself has this property. This follows immediately from the fact that if an excess return $r^{(A)}$ first or second order stochastically dominates an excess return $r^{(B)}$, then the expected value of $r^{(A)}$ is at least as large as that of $r^{(B)}$.[3] In other words, if $r^{(A)} \overset{①}{\succ} r^{(B)}$ or $r^{(A)} \overset{②}{\succ} r^{(B)}$, then $\mathsf{AS}\left(r^{(A)}\right) < \mathsf{AS}\left(r^{(B)}\right)$ and $\mathbb{E}\left(r^{(A)}\right) \geq \mathbb{E}\left(r^{(B)}\right)$ and therefore $\mathsf{EPM}\left(r^{(A)}\right) > \mathsf{EPM}\left(r^{(B)}\right)$.

Normally distributed returns

For normally distributed excess returns ($r^{(N)} \sim \mathcal{N}(\mu, \sigma^2)$) the EPM is given by[4]

$$\mathsf{EPM}\left(r^{(N)}\right) = \frac{\mu}{AS(r^{(N)})} = \frac{2\mu^2}{\sigma^2}.$$

In this case the EPM equals two times the squared Sharpe ratio, $\mathsf{EPM}\left(r^{(N)}\right) = 2\mathsf{SR}^2\left(r^{(N)}\right)$,

[2] To be more explicit, let W_0 denote initial wealth and let $\{r_n\}_{n\geq 1}$ a sequence of independent and identically distributed excess returns, for which the AS index exists. Wealth at time n is given by $W_n = W_0 + r_1 + r_2 + \ldots + r_n$. Bankruptcy occurs as soon as $W_n < 0$.

[3] To see this, note that $u(x) = x$ is increasing and concave and thus "$r^{(A)} \overset{①}{\succ} r^{(B)}$" or "$r^{(A)} \overset{②}{\succ} r^{(B)}$" implies $\mathbb{E}\left(r^{(A)}\right) = \mathbb{E}\left(u(r^{(A)})\right) \geq \mathbb{E}\left(u(r^{(B)})\right) = \mathbb{E}\left(r^{(B)}\right)$.

[4] This is readily inferred from the fact the the moment generating function of a normally distributed random variable with mean μ and variance σ^2 is given by $e^{\mu t + \sigma^2 t^2/2}$.

which means that the two measures produce identical rankings. Hence, for normally distributed excess returns, where the Sharpe ratio is an appropriate performance measure, the EPM suits equally well. But the EPM is also suitable, when other moments of the distribution, such as skewness and kurtosis, are important. In the next paragraph we supplement this result and show that, under some regularity conditions, as the distribution of an excess return r approximates the normal distribution, the EPM of r approximates the EPM of a normally distributed excess return.

Generalized continuity

We show that the continuity property of the AS index also applies to more general cases than those considered by Aumann and Serrano (2008). From this and an additional assumption the continuity of the EPM follows. The above mentioned result concerning the approximation of normally distributed returns is a special case of this *generalized continuity* property of the EPM.

Let r_0 and $\{r_n\}_{n\geq 1}$ be random variables and denote their moment generating function as $M_n(t) = \mathbb{E}\left(e^{tr_n}\right)$ $(n \geq 0)$. Let "$\xrightarrow{d}$" denote convergence in distribution.

Assumption 2.1. *For all $n \geq 0$ the economic index of riskiness $\mathsf{AS}\left(r_n\right)$ exists.*

Assumption 2.2. *There exists a $b > \mathsf{AS}\left(r_0\right)$ for which $\sup_n M_n(-b) < \infty$.*

Assumption 2.2 means that the left tails of the return distributions should not be too heavy. We these two assumptions we can state

Proposition 2.3 (Generalized Continuity)**.** *If Assumptions 2.1 and 2.2 hold, then $r_n \xrightarrow{d} r_0$ implies $\mathsf{AS}\left(r_n\right) \to \mathsf{AS}\left(r_0\right)$.*

Proof. The proof of Proposition 2.3 is given in Appendix 6.1. As the main tool we use Theorem 5.4 in Billingsley (1968). □

Remark 2.4. Aumann and Serrano (2008) assume that the excess returns satisfy conditions (i) and (ii), take only finitely many values, and that they be uniformly bounded[5]. These assumptions are stronger and indeed imply Assumptions 2.1 and 2.2. First, requiring that the returns take on only finitely many values and satisfy conditions *(i)* and *(ii)* assures that the AS index exists for the returns under consideration. Second, requiring that the sequence of returns be uniformly bounded implies Assumption 2.2. To see this, let $K > 0$ such that $|r_n| \leq K$ for all n. Then, for $b > \mathsf{AS}\left(r_0\right)$, $e^{-br_n} \leq e^{bK}$ almost surely for all n. Therefore, $M_n(-b) = \mathbb{E}\left(e^{-br_n}\right) \leq e^{bK}$ for all n and Assumption 2.2 holds.

[5]A sequence of excess returns $\{r_n\}_{n\geq 1}$ is uniformly bounded, if there is a $K > 0$ such that $|r_n| \leq K$ almost surely for all n.

So far we have proven *generalized continuity* for the AS index. To obtain continuity for the EPM we have to make an additional assumption:

Assumption 2.5. *The sequence $\{r_n\}_{n\geq 1}$ is uniformly integrable.*[6]

Corollary 2.6 (*Generalized Continuity for EPM*)**.** *If Assumptions 2.1, 2.2, and 2.5 hold, then $r_n \xrightarrow{d} r_0$ implies $\mathsf{EPM}(r_n) \to \mathsf{EPM}(r_0)$.*

Assumption 2.5 together with $r_n \xrightarrow{d} r_0$ implies that $\mathbb{E}(r_n) \to \mathbb{E}(r_0)$ (cf. Billingsley, 1968). Then the corollary immediately follows from *generalized continuity*. Furthermore, note that the requirement of $\{r_n\}_{n\geq 1}$ being uniformly bounded, as in Aumann and Serrano (2008), guarantees that Assumption 2.5 holds.

This property is not only of theoretical importance but also of practical relevance. Note, that the aggregational Gaussianity property of (excess) asset returns states that the distribution of less frequently sampled returns, e.g. going from daily to monthly and down to yearly returns, approximates the normal distribution. Under the assumptions of this section, this implies that the EPM approximates the EPM of normally distributed returns, as the sampling frequency decreases. While the Sharpe ratio is appropriate for low frequency returns, the EPM is appropriate for both low and high frequency returns, with no disadvantages compared to the Sharpe ratio in the former case.

2.3 Relation to the Sharpe ratio and other performance measures

The EPM belongs to the general class of performance measures which consider risk-adjusted returns

$$\mathsf{RAR}(\tilde{r};\theta) = \begin{cases} \frac{m(\tilde{r};\theta)}{s(\tilde{r};\theta)} & \text{if a specific condition is satisfied, e.g. } \mathbb{E}(\tilde{r}) > r_f, \\ 0 & \text{else,} \end{cases} \tag{2.3}$$

with m a reward measure, s a risk measure and θ a vector of parameters, e.g. the risk free rate r^f . The most prominent member of this class of performance measures is the Sharpe ratio given by

$$\mathsf{SR}(r) = \frac{\mathbb{E}(\tilde{r}) - r^f}{\sqrt{\mathbb{V}(\tilde{r})}},$$

which is sensible if $\mathbb{E}(\tilde{r}) > r^f$. The Sharpe ratio plays an important role in a mean-variance decision framework. There, the portfolio with the highest Sharpe ratio together with the risk free asset determine the set of (mean-variance) efficient investments, e.g. see Sharpe (1966) and Treynor (1965). Outside the mean-variance framework, however, the Sharpe

[6]By definition, this means that $\lim_{\alpha\to\infty} \sup_n \mathbb{E}\left(|r_n| \mathbb{I}_{(|r_n|\geq\alpha)}\right) = 0$.

ratio ignores potentially important aspects of return distributions such as skewness and kurtosis.

In response to the drawbacks of the standard deviation as risk measure several other performance measures, that fall into the category of reward-risk ratios, have been proposed. These *reward-risk* performance ratios use lower partial moments[7] or value at risk (VaR) in the denominator. For illustration and to focus the following discussion we present some examples.

Arguing that investors care especially about downside dispersion Sortino and Price (1994) proposed to use

$$\mathsf{SR}\left(\tilde{r};\lambda\right)=\begin{cases}\frac{\mathbb{E}(\tilde{r})-\lambda}{\sqrt{\mathbb{E}\left((\lambda-\tilde{r})^2\mathbb{I}_{(-\infty,\lambda)}(\tilde{r})\right)}} & \text{if } \mathbb{E}\left(\tilde{r}\right)>\lambda,\\ 0 & \text{else,}\end{cases}$$

as performance measure. The threshold-parameter $\lambda > 0$ divides return movements into desirable (relative) upward moves and undesirable downward moves and can be interpreted as a *minimal acceptable return*. Examples are the risk-free return or the inflation rate.

Although it has not been introduced as a performance measure, the *gain-loss ratio* of Bernardo and Ledoit (2000) is often used to measure risk-adjusted return. Letting $r^+ = \max(r,0)$ denote the positive part of the excess return and $r^- = -\min(r,0)$ the negative part, the *gain-loss ratio* is given by

$$\mathsf{GL}\left(\tilde{r};r^f\right)=\frac{\mathbb{E}\left(r^+\right)}{\mathbb{E}\left(r^-\right)}=\frac{\mathbb{E}\left(\left(\tilde{r}-r^f\right)^+\right)}{\mathbb{E}\left(\left(\tilde{r}-r^f\right)^-\right)}.$$

Due to the regulatory requirements of Basle II the VaR has become popular as a risk measure. For an excess return $\tilde{r}-r^f$, $\mathsf{VaR}_\alpha\left(\tilde{r}-r^f\right)$ is the threshold that negative excess returns will not exceed with a given probability α.[8] The corresponding performance measure, termed as *excess return on VaR* (cf. Dowd, 2000), is given by

$$\mathsf{ERVaR}_\alpha\left(\tilde{r};r^f\right)=\begin{cases}\frac{\mathbb{E}(\tilde{r})-r^f}{\mathsf{VaR}_\alpha\left(\tilde{r}-r^f\right)} & \text{if } \mathbb{E}\left(r\right)>r^f,\\ 0 & \text{else.}\end{cases}$$

Discussion

The EPM has several advantages over the other performance measures. In particular, the EPM is (strictly) monotone with respect to stochastic dominance . Before we illustrate

[7] For $n \in \mathbb{N}_0$ we define the n-th partial moment of a random variable X with truncation parameter $\lambda \in \mathbb{R}$ as: $\mathbb{E}\left((\lambda-X)^n\mathbb{I}_{(\lambda,\infty)}\left(X\right)\right)$

[8] Formally, $\mathsf{VaR}_\alpha\left(\tilde{r}-r^f\right)=-F^{-1}_{\tilde{r}-r^f}\left(1-\alpha\right)$, where $F^{-1}_{\tilde{r}-r^f}$ denotes the inverse distribution function of $\tilde{r}-r^f$.

this by means of an example, we will briefly present the concept of stochastic dominance.[9] Thereby, it will become apparent, that monotonicity with respect to stochastic dominance is quite an important property for a performance measure.

In the theory of choice under uncertainty, stochastic dominance is a widely acknowledged concept. If an investment A stochastically dominates an investment B, then a "large" group of investors will prefer A over B. More specifically, let F_A and F_B be the distribution functions of the excess returns $r^{(A)}$ and $r^{(B)}$ corresponding to A and B, respectively. A first order stochastically dominates B ($A \overset{①}{\succ} B$), if $F_A(x) \leq F_B(x)$ for all $x \in \mathbb{R}$ and $F_A(x) < F_B(x)$ for some $x \in \mathbb{R}$. If this is the case, then any decision maker with increasing utility function will prefer investment A over investment B. Moreover, the converse is also true. Unfortunately, many investments cannot be ordered by first order stochastic dominance. A less restrictive assumption about the relation of two gambles is made by second order stochastic dominance ($\overset{②}{\succ}$). We say that an investment A second order stochastically dominates an investment B ($A \overset{②}{\succ} B$), if $\int_{-\infty}^{y} [F_A(x) - F_B(x)] \,\mathrm{d}\, x \leq 0$ for all $y \in \mathbb{R}$ and strict inequality holds for some $y \in \mathbb{R}$. Note that second order stochastic dominance is implied by first order stochastic dominance. Similar to first order stochastic dominance, it holds that, $A \overset{②}{\succ} B$ *iff* any investor with increasing and concave utility function prefers A to B.

It is plausible to require from a performance measure that it assesses A as less risky than B, if all profit seeking and risk averse investors would prefer A over B (i.e. $A \overset{②}{\succ} B$). However the risk-adjusted return measures from this section are not (strictly) monotone with respect to second order stochastic dominance. As an example consider an investment A with excess returns -10% or 20% each with probability $1/2$ and an investment B with $\mathbb{P}(r^{(B)} = -10\%) = 0.5$, $\mathbb{P}(r^{(B)} = 10\%) = 0.25$, and $\mathbb{P}(r^{(B)} = 30\%) = 0.25$. It is readily verified, that the Sortino ratio (for $\lambda \leq 1$), the gain-loss ratio, and the excess return on VaR (for reasonable probabilities α) each assign the same value to both A and B, although $A \overset{②}{\succ} B$. Thus, these measures ignore a change in risk, which no (strictly) risk averse investor would be indifferent to.

The Sharpe ratio does not even respect first order stochastic dominance. For example, consider two assets A and B, where A yields an excess return $r^{(A)}$ of either 1% or 5%, both with equal probability 0.5, and B yields an excess return $r^{(B)}$ of either 1% or 3%, also both with probability 0.5. Then, $\mathsf{SR}\left(r^{(A)}\right) = 1.5 < 2 = \mathsf{SR}\left(r^{(B)}\right)$, although it would be natural to prefer A over B, since asset A performs as well as asset B in one case and strictly better than B in the other.

Besides the fact that the EPM is free from these disadvantages, it also uses an economically well-founded risk measure. In fact, the AS index is based on two axioms, which uniquely

[9] We leave out some cumbersome technicalities. For details see Tesfatsion (1976).

characterize the index up to a positive factor. Artzner, Delbeaen, Eber, and Heath (1999) also use an axiomatic approach, but their axioms give rise to a whole class of *coherent risk measures* (instead of determining one) which are not necessarily monotone with respect to stochastic dominance. For further discussion we refer to Aumann and Serrano (2008).

3 Estimation of the economic performance measure

In empirical applications the probability measure $\mathbb{P}$ is not observable but instead we observe n independent and identically distributed (iid) realizations $\{r_1, \ldots, r_n\}$ of the excess return r.[10] Based on these observations we want to estimate the EPM, the ratio of the mean of the excess returns and the corresponding AS index of riskiness. As the estimation of the mean is straight forward we will concentrate on the estimation of the AS index.

3.1 Parametric estimation and the normal inverse Gaussian distribution

A typical parametric estimation approach consists in choosing a reasonable parametric distribution, estimating the parameters of this distribution and, finally, computing the AS index based on these estimates. We suggest to use the normal inverse Gaussian (NIG) distribution for parametric estimation. The NIG distribution is a well established distribution in finance, econometrics and statistics. It is used for example to model unconditional as well conditional return distributions, see e.g. Andersson (2001); Bollerslev, Kretschmer, Pigorsch, and Tauchen (2009) as well as Eriksson, Ghysels, and Wang (2009). We derive the AS index and the EPM for NIG-distributed returns. Our representation of the EPM in terms of the mean, variance, skewness, and excess kurtosis offers a simple estimation scheme for the EPM. Furthermore, it makes explicit the role of higher order moments, which are neglected in the Sharpe ratio.

A NIG-distributed excess return (random variable) $r^{(NIG)} \sim \mathcal{NIG}(\alpha, \beta, \nu, \delta)$ is characterized by the following density

$$f^{(NIG)}(x; \alpha, \beta, \nu, \delta) = \frac{\alpha\delta}{\pi} \frac{K_1\left(\alpha\sqrt{\delta^2 + (x-\nu)^2}\right)}{\sqrt{\delta^2 + (x-\nu)^2}} \, \mathrm{e}^{\delta\gamma + \beta(x-\nu)} \tag{3.1}$$

with $0 \leq |\beta| < \alpha$, $\delta > 0$, $\nu \in \mathbb{R}$, $\gamma = \sqrt{\alpha^2 - \beta^2}$ and $K_1(y) = (1/2) \int_0^\infty \mathrm{e}^{-(1/2)y(z+z^{-1})} \, \mathrm{d}\, z$ the modified Bessel function of the third kind with index 1. δ is a scaling parameter, ν is

[10] Although we assume iid random variables in the following this is not essential and the estimator can be straightforwardly generalized to a more general dependence structure.

a location parameter, β is an asymmetry parameter and $\alpha \pm \beta$ determines the heaviness of the tails. It follows from (3.1) that the moment generating function is given by

$$\mathbb{E}\left(\mathrm{e}^{tr^{(NIG)}}\right) = M^{(NIG)} : [-(\alpha+\beta), \alpha-\beta] \to \mathbb{R} \quad t \mapsto \mathrm{e}^{\nu t+\delta(\gamma-\sqrt{\alpha^2-(\beta+t)^2})}. \tag{3.2}$$

For the derivation of the EPM and the AS index we first have to ensure that the latter is well defined for NIG-distributed gambles. Note that the two assumptions (i) and (ii) are obviously satisfied whenever the mean $(\nu + \delta\beta/\gamma)$ is positive. However, (i) and (ii) are not sufficient for the existence of the risk index. For a NIG-distributed random variable condition (iii) is satisfied, if $\nu \leq \delta(\alpha-\beta)/\gamma$. Using the defining Equation (2.2) and Equation (3.2) the AS index and the EPM for NIG-distributed gambles result as

$$\mathsf{AS}^{(NIG)}(\alpha,\beta,\nu,\delta) = \mathsf{AS}\left(r^{(NIG)}\right) = \frac{1}{2}\frac{\delta^2+\nu^2}{\beta\delta^2+\gamma\delta\nu} \tag{3.3}$$

$$\mathsf{EPM}^{(NIG)}(\alpha,\beta,\nu,\delta) = \mathsf{EPM}\left(r^{(NIG)}\right) = \frac{2(\nu+\delta\beta/\gamma)(\beta\delta^2+\gamma\delta\nu)}{\delta^2+\nu^2}. \tag{3.4}$$

Details of the derivation are provided in Appendix 6.2. Due to the parameter restrictions implied by condition (ii) the AS index is always positive, as it should be. It also can be easily verified that $-1/\mathsf{AS}^{(NIG)}(\alpha,\beta,\nu,\delta)$ is in the domain of $M^{(NIG)}$.

A representation of the AS index in terms of the moments is given by:

$$\widetilde{\mathsf{AS}}^{(NIG)}\left(\mu,\sigma^2,\chi,\kappa\right) = \left(3\kappa\mu - 4\mu\chi^2 - 6\chi\sigma + 9\sigma^2/\mu\right)/18 \tag{3.5}$$

$$\widetilde{\mathsf{EPM}}^{(NIG)}(m,v,s,k) = 18\mu/\left(3\kappa\mu - 4\mu\chi^2 - 6\chi\sigma + 9\sigma^2/\mu\right) \tag{3.6}$$

with mean $\mu > 0$, variance $\sigma^2 > 0$, excess kurtosis $\kappa > 0$ and skewness $|\chi| < \sqrt{3\kappa/5}$. The derivation is provided in Appendix 6.3. Note that assumption (iii) can also be rewritten in terms of those moments as $\mu \leq 3\sqrt{\sigma^2/(3\kappa - 4\chi^2)}$.

An obvious and efficient way to obtain parameter estimates is maximum likelihood estimation. Then, (3.4) can be used to compute the EPM. However, we have a computationally less expensive method available. A very practical way is to use the representation in (3.6) and compute the EPM using empirical moments.

Remark 3.1. For $\chi = 0$ and $\kappa \to 0$ in (3.6), we obtain the EPM for a normally distributed gamble with mean μ and variance σ^2: $(2\mu^2)/\sigma^2$. Thus, the order induced by the EPM for NIG-distributed returns approximates that induced by the Sharpe ratio if skewness and excess kurtosis go to zero. As opposed to the Sharpe ratio, $\widetilde{\mathsf{EPM}}^{(NIG)}$ can easily account for skewness and kurtosis in cases where these are not negligible.

3.2 Nonparametric estimation

The choice of a reasonable parametric distribution is not always obvious and in certain cases one might prefer a non-parametric approach to estimate the AS index of riskiness. A completely natural way to estimate the AS index is by Method of Moments (MM) (see Hansen, 1982). We set

$$f(x; s) = \mathrm{e}^{-x/s} - 1 \quad (s > 0)$$

and consider the moment equation

$$\mathbb{E}\left(f(r; s_0)\right) = \mathbb{E}\left(\mathrm{e}^{-r/s_0} - 1\right) = 0 \quad (s_0 > 0), \tag{3.7}$$

where r is excess return/ random variable underlying the realizations $\{r_1, \ldots, r_n\}$. Note that (3.7) corresponds to the defining equation for the AS index (2.2), i.e. $s_0 = AS(r)$. The MM-estimator $\hat{s}_n$, where we use the subscript n to exhibit the dependence on the sample size, is given by the solution to the empirical counterpart of (3.7):

$$\frac{1}{n}\sum_{i=1}^{n} \mathrm{e}^{-r_i/s} - 1 = 0. \tag{3.8}$$

Equation (3.8) has to be solved numerically. A solution can always be found, if some of the realizations r_i are negative and $\frac{1}{n}\sum_{i=1}^{n} r_i > 0$. By the strong law of large numbers, this will almost surely be the case, for a large sample size n, if the generic excess return r satisfies conditions (i) and (ii) in Section 2.1.

Within the MM setup for uncorrelated random variables the asymptotic distribution of the estimator is given by

$$\sqrt{n}\left(\hat{s}_n - s_0\right) \xrightarrow{\mathscr{D}} \mathcal{N}\left(0, \frac{S}{G_0^2}\right) \tag{3.9}$$

where

$$G_0 = \mathbb{E}\left(\left.\frac{\partial f(g; s)}{\partial s}\right|_{s=s_0}\right) = \frac{1}{s_0^2}\mathbb{E}\left(\mathrm{e}^{-r/s_0} r\right) \tag{3.10}$$

and

$$S = \mathbb{V}\left(f(r; s_0)\right) = \mathbb{E}\left(f(r; s_0)^2\right) = \mathbb{E}\left(\mathrm{e}^{-2r/s_0}\right) - 1 \tag{3.11}$$

where we have assumed the existence of the necessary moments, i.e. the mgf of r has to exists at $-2/s_0$ (see (3.11)): The variance of $\hat{s}_n$ can be estimated using the empirical counterparts of G_0 and S and replacing s_0 with $\hat{s}_n$.

Some insight can be gained by looking at the above estimation procedure from a different perspective. Given realizations $\{r_1, \ldots, r_n\}$ of the random excess return r one can define the *empirical distribution function*

$$\hat{F}_n(x) = \sum_{i=1}^{n} \mathbb{I}_{(-\infty,x]}(r_i)\,. \tag{3.12}$$

$\hat{F}_n(x)$ is the distribution of a gamble that takes on the values $\{r_1, \ldots, r_n\}$ each with probability $1/n$. The mgf pertaining to $\hat{F}_n(x)$ is the *empirical mgf* and is given by

$$\hat{M}_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{e}^{tr_i}\,. \tag{3.13}$$

By the Glivenko-Cantelli theorem, $\hat{F}_n(x)$ converges uniformly to the distribution function of r.[11] Therefore, one can hope that solving $\hat{M}_n(-1/s) = 1\,(s > 0)$ will yield a good estimate of the AS index of r. In fact, this estimate is exactly $\hat{s}_n$.

Remark 3.2. There is an interesting analogy to expected utility theory where mean-variance analysis is justified by assuming either normal returns or that the utility function can be reasonably well approximated by second order Taylor-expansion. In fact, instead of solving (3.8) one could equally well solve $h(t) = \log[\hat{M}_n(t)] = 0$ $(t < 0)$. Denoting the solution as t^*, $\hat{s}_n$ is given by $-1/t^*$. The second order Taylor-expansion of $h(t)$ can be written as

$$\hat{h}(t) = \hat{\mu}_r t + \frac{1}{2}\hat{\sigma}_r^2 t^2,$$

where $\hat{\mu}_r$ and $\hat{\sigma}_r^2$ are the empirical mean and variance of r. The solution of $\hat{h}(t) = 0$ $(t < 0)$ leads to an estimate $\hat{\sigma}_r^2/(2\hat{\mu}_r)$ for the AS index and $2\hat{\mu}_r^2/\hat{\sigma}_r^2$ for the EPM. The latter is the empirical counterpart of the EPM for normal returns. Thus, similar to truncating the expansion of the utility function, approximating $h(t)$ by a second order Taylor-expansion leads back to the mean-variance decision framework. Second order Taylor's series approximations, however, risk a serious loss of accuracy (cf. Loistl (1976)).

Remark 3.3. There is a remarkable relation between the AS index and the so called *adjustment coefficient* (AC) from ruin theory. In fact one is the reciprocal of the other

$$AS = \frac{1}{AC}. \tag{3.14}$$

Pitts, Grübel, and Embrechts (1996) proposed to estimate the adjustment coefficient as the solution of $\hat{M}(-s) = 1\,(s > 0)$, which ends up being the reciprocal of $\hat{s}_n$. Without referring to the methods of moments, they showed that their estimator is asymptotically normal with

[11] Uniform convergence of a function F_n to F means that $\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \to 0$ as $n \to \infty$.

rate $\sqrt{n}$ and asymptotic variance

$$\frac{M_g(-2AC)-1}{M_g'(-AC)^2},$$

provided that the mgf of g, M_g, exists at $-2AC$. Using the reciprocal relation between the AS index and the adjustment coefficient and applying the so-called Delta-method this could also have been derived from (3.9–3.11), or vice versa.

Remark 3.4. Consistency of $\hat{s}_n$ as estimator for the AS index is also obtained by using *generalized continuity*. For this, the requirement that the mgf of r exist at $-2/AS$ can be replaced by the weaker requirement that M_r exists at some b smaller than $-1/AS$. Consider the sequence of gambles $f_1, f_2, \ldots$, where f_n follows the distribution given by $\hat{F}_n(x)$ in (3.12) for realization $\{r_1, \ldots, r_n\}$. Almost surely, the AS index of f_n exists for large n and equals $\hat{s}_n$ (assuming (i) and (ii) in Section 2.1 hold for r). Moreover, since M_r exists at some b smaller than $-1/AS$, e^{br} is integrable. Then, by the the strong law of large numbers, the mgf of f_n at b, $\hat{M}_n(b)$, converges to $M_r(b)$ almost surely. This implies that the sequence $\{\hat{M}_n(b)\}_{n\geq 1}$ is (almost surely) bounded. Since, by the Glivenko-Cantelli theorem, $f_n \xrightarrow{d} r$, we can apply *generalized continuity* and conclude that $\hat{s}_n \to s_0 = AS(g)$ almost surely.

4 Empirical illustration

We consider the 30 largest-growth mutual funds (as of January 1998 in terms of overall assets managed, cf. Vinod and Morey (2001)) and use monthly excess returns from January 1987 to 2008. The excess returns are computed from monthly fund returns and the one-months US treasury bill rate.

The resulting estimates for the performance measures are reported in Table 4. The ranking generated by the respective performance measure is given in brackets. The columns labeled EPMNon and EPMNIG display the values for the non-parametric and the parametric estimates using the NIG-distribution, respectively. The table also shows the empirical skewness and kurtosis of the funds together with the ranking according to these measures.[12]

Note that the rankings induced by the two estimators for the EPM are very similar and in most cases identical. This is also supported by the rank correlation coefficient (Kendall's τ) which equals 0.9862 (cf. Table 2).[13] The rank correlation of the Sharpe ratio ranking with either of the EPM rankings is considerably lower. Since the Sharpe ratio neglects skewness

[12]We consider low kurtosis to be preferable to high kurtosis and high skewness to be preferable to low skewness.

[13]Kendall's τ equals 1 if two rankings perfectly agree, 0 if they are independent, and -1 if they perfectly disagree.

and kurtosis, these measures can serve to explain the difference between the Sharpe ratio ranking and the EPM rankings. In general a fund's EPM ranking deteriorates relative to the Sharpe ratio ranking, if the fund's skewness is low and/ or its kurtosis is high, and vice versa. For instance *Fidelity Contrafund* ranks number 1 according to the Sharpe ratio, but only number 4 (5) according to the EPM, which is probably due to its low ranking in skewness (29) and kurtosis (30). On the other hand *IDS New Dimensions A* displays high skewness and low kurtosis and ranks second instead of fifth when EPM ranking instead of Sharpe ratio ranking is used.

Table 1: **Performance measures**

name	Sharpe	EPMNon	EPMNIG	skewness	kurtosis
AIM Value A	0.2462 (4)	0.0975 (5)	0.0960 (4)	-1.1797 (17)	8.3084 (16)
AIM Weingarten A	0.1897 (22)	0.0643 (17)	0.0638 (17)	-0.6431 (6)	7.4478 (11)
Amcap	0.1868 (23)	0.0637 (19)	0.0632 (18)	-0.5755 (3)	6.2021 (4)
American Cent-Growth	0.1573 (28)	0.0454 (28)	0.0451 (28)	-0.6096 (5)	7.4150 (10)
American Cent-Select	0.1554 (29)	0.0429 (29)	0.0425 (29)	-0.9755 (13)	8.8714 (17)
Brandywine	0.1909 (19)	0.0605 (21)	0.0596 (21)	-1.2366 (18)	10.3511 (20)
Davis NY Venture A	0.2556 (2)	0.1113 (1)	0.1100 (1)	-0.7710 (8)	6.3943 (6)
Fidelity Contrafund	0.2690 (1)	0.0980 (4)	0.0926 (5)	-2.0853 (29)	16.6301 (30)
Fidelity Destiny I	0.2426 (6)	0.0918 (6)	0.0898 (6)	-1.3421 (20)	10.1994 (19)
Fidelity Destiny II	0.2425 (7)	0.0909 (7)	0.0888 (8)	-1.4108 (22)	10.5906 (22)
Fidelity Growth	0.1996 (13)	0.0679 (13)	0.0672 (13)	-1.0125 (15)	8.2124 (15)
Fidelity Magellan	0.2151 (10)	0.0731 (11)	0.0717 (11)	-1.4733 (24)	11.0436 (24)
Fidelity OTC	0.1899 (21)	0.0567 (24)	0.0557 (24)	-1.7688 (27)	13.1197 (26)
Fidelity Ret. Growth	0.1783 (25)	0.0520 (26)	0.0512 (26)	-1.5171 (25)	11.9384 (25)
Fidelity Trend	0.1335 (30)	0.0311 (30)	0.0308 (30)	-1.4335 (23)	10.6231 (23)
Fidelity Value	0.1941 (17)	0.0575 (23)	0.0563 (23)	-2.1504 (30)	13.8901 (28)
IDS Growth A	0.1907 (20)	0.0659 (15)	0.0655 (15)	-0.6703 (7)	5.7381 (3)
IDS New Dimensions A	0.2433 (5)	0.1092 (2)	0.1085 (2)	-0.3113 (2)	5.1745 (2)
Janus	0.2484 (3)	0.1081 (3)	0.1067 (3)	-0.5956 (4)	6.2798 (5)
Janus Twenty	0.2073 (11)	0.0742 (10)	0.0734 (10)	-0.8840 (10)	7.5067 (12)
Legg Mason Value Prim	0.2020 (12)	0.0675 (14)	0.0666 (14)	-1.3083 (19)	8.9115 (18)
Neuberger & Ber Part	0.2308 (8)	0.0903 (8)	0.0894 (7)	-0.9882 (14)	6.5216 (7)
New Economy	0.1923 (18)	0.0638 (18)	0.0631 (19)	-1.0632 (16)	7.3312 (9)
Nicholas	0.2283 (9)	0.0887 (9)	0.0877 (9)	-0.9398 (11)	6.8092 (8)
PBHG Growth PBHG	0.1774 (26)	0.0605 (20)	0.0601 (20)	-0.3022 (1)	4.1678 (1)
Prudential Equity B	0.1983 (15)	0.0600 (22)	0.0586 (22)	-1.9457 (28)	14.5752 (29)
T. Rowe Price Growth	0.1817 (24)	0.0548 (25)	0.0540 (25)	-1.3852 (21)	10.4853 (21)
Van Kampen Pace	0.1688 (27)	0.0464 (27)	0.0456 (27)	-1.6258 (26)	13.5125 (27)
Vanguard U.S. Growth	0.1983 (14)	0.0687 (12)	0.0680 (12)	-0.7779 (9)	7.9925 (13)
Vanguard/Primecap	0.1943 (16)	0.0651 (16)	0.0644 (16)	-0.9573 (12)	8.0177 (14)

Table 2: **Rank correlation (Kendall)**

1.0000	0.8437	0.8299
0.8437	1.0000	0.9862
0.8299	0.9862	1.0000

5 Conclusion

In this paper we propose a new performance measure (EPM) that generalizes the Sharpe ratio. The EPM employs as risk measure the Aumann-Serrano (2008) index of riskiness instead of the standard deviation. If returns are normally distributed, the EPM and the Sharpe ratio induce equivalent rankings. Contrary to the Sharpe ratio, the EPM respects stochastic dominance and accounts for skewness and kurtosis in the return distribution. As part of the *generalized continuity* property of the EPM, the EPM converges to two times the squared Sharpe ratio, if the distribution of the returns converges to the normal distribution. In this sense, the EPM is asymptotically equivalent to the Sharp ratio.

We have calculated the EPM for returns that follow a normal inverse Gaussian (NIG) distribution, a distribution that is well suited to model financial returns. A representation of the EPM for NIG-distributed returns in terms of the first four moments makes explicit how skewness and kurtosis enter the performance measure. If these two moments go to zero, the EPM again converges to two times the squared Sharpe ratio. If skewness and kurtosis are not negligible, however, the EPM corrects the Sharpe ratio for theses two moments. Furthermore, using the NIG-distribution provides a parametric way to estimate the EPM, which is virtually as easy estimating the Sharpe ratio with empirical moments.

For non-parametric estimation of the EPM we suggest to use the method of moments (cf. Hansen (1982)) with the defining equation of the Aumann-Serrano index as moment equation. This moment equation can also be represented by using the empirical cumulant generating function. The approximation of the cumulant generating function by second order Taylor-expansion results in an EPM equal to two times the squared Sharpe ratio and thus leads back to the mean-variance decision framework, with a risk of loosing precision.

As empirical illustration we rank mutual funds with the Sharpe ratio and the EPM. The results show that the EPM penalizes investments with significant excess kurtosis (or negative skewness), while the Sharpe ratio neglects these features.

Appendix

6 Proofs and Derivations

6.1 Proof of Proposition 2.3

Let r_0 and $\{r_n\}_{n\geq 1}$ be random variables with moment generating function $M_n(t) = \mathbb{E}\left(\mathrm{e}^{tr_n}\right)$ $(n \geq 0)$. Assume that

- for all $n \geq 0$ the economic index of riskiness $AS(r_n)$ exists,
- there exists a $b > AS(r_0) > 0$ for which $sup_n M_n(-b) < \infty$, and
- $r_n \xrightarrow{d} r_0$

We want to show, that $AS(r_n) \rightarrow AS(r_0)$.

Proof: From $r_n \xrightarrow{d} r_0$ and the continuous mapping theorem it follows that $\mathrm{e}^{-tr_n} \xrightarrow{d} \mathrm{e}^{-tr_0}$ for all $t \in [0, b]$. Moreover, for $t \in (0, b)$ there exists $\epsilon > 0$ such that $(1 + \epsilon)t = b$. Therefore,

$$\begin{aligned} & \sup_n \mathbb{E}\left([\mathrm{e}^{-tr_n}]^{1+\epsilon}\right) = \sup_n \mathbb{E}\left(\mathrm{e}^{-br_n}\right) = \sup_n M_n(-b) < \infty \\ \Rightarrow \quad & \{\mathrm{e}^{-tr_n}\} \text{ is uniformly integrable} \\ \Rightarrow \quad & \mathbb{E}\left(\mathrm{e}^{-tr_n}\right) \rightarrow \mathbb{E}\left(\mathrm{e}^{-tr_0}\right) \qquad \text{(cf. Theorem 5.4 in Billingsley (1969).} \end{aligned}$$

So far we have shown that $M_n(-t) \rightarrow M_0(-t)$ pointwise in $[0, b)$. To see that this implies $AS(r_n) \rightarrow AS(r_0)$, define $L_n(t) \equiv M_n(-t)$ and $\alpha_n \equiv 1/AS(r_n)$ for $n \geq 0$. It suffices to show that $\alpha_n \rightarrow \alpha_0$. Observe that the following holds for all $n \geq 0$:

- $L_n(0) = 1$
- $L_n'(0) = -\mathbb{E}\left(r_n\right) < 0$ (where $L_n'(0)$ is the derivative of $L_n$evaluated at 0)
- $L_n''(t) = \mathbb{E}\left(r_n^2\,\mathrm{e}^{-tr_n}\right) \geq 0$

The second point is true, since $\mathbb{E}\left(r_n\right) > 0$ is necessary for the existence of the AS index. So for each n, $L_n(\cdot)$ is convex. It takes on the value 1 at 0 and then initially decreases. Since $L_n(\alpha_n) = 1$, L_n must be strictly increasing in a neighborhood of α_n. Now let $\epsilon > 0$ arbitrary. We can assume that $[\alpha_0 - \epsilon, \alpha_0 + \epsilon] \subseteq (0, b)$. Furthermore,

$$L_0(\alpha_0) = 1 \qquad \Rightarrow \qquad L_0(\alpha_0 - \epsilon) < 1 \text{ and } L_0(\alpha_0 + \epsilon) > 1$$

Because of (pointwise) convergence, we can find a $N(\epsilon) \in \mathbb{N}_0$ such that

$$L_n(\alpha_0 - \epsilon) < 1 \text{ and } L_n(\alpha_0 + \epsilon) > 1 \quad \text{for all } n \geq N(\epsilon).$$

But, by the properties of the the functions L_n, this implies

$$\alpha_n \in (\alpha_0 - \epsilon, \alpha_0 + \epsilon) \quad \text{for all } n \geq N(\epsilon)$$

q.e.d.

6.2 Derivation of the AS index for NIG-distributed random variables

Let X be a random variable following the normal inverse Gaussian (NIG) distribution with parameters α, β, ν and δ where $0 \leq |\beta| < \alpha$, $\delta > 0$ and $\nu \in \mathbb{R}$. The unique Aumann–Serrano risk index $s > 0$ (if it exists) is implicitly defined as

$$\mathbb{E}\left(\mathrm{e}^{-X/s}\right) = 1. \tag{6.1}$$

Let M_X denote the moment generating function (mgf) of X given by (3.2). Then (6.1) can be equivalently written as

$$M_X(t_s) = 1 \tag{6.2}$$

with $t_s = -1/s$. Solving for t_s we obtain

$$\begin{aligned}
&\mathrm{e}^{\nu t + \delta\left(\gamma - \sqrt{\alpha^2 - (\beta+t)^2}\right)} = 1 \\
\Leftrightarrow\ & \nu t + \delta\left(\gamma - \sqrt{\alpha^2 - (\beta+t)^2}\right) = 0 \\
\Leftrightarrow\ & \nu t + \delta\gamma = \delta\sqrt{\alpha^2 - (\beta+t)^2} \\
\Rightarrow\ & \nu^2 t^2 + 2\delta\gamma\nu t + \delta^2\gamma^2 = \delta^2\alpha^2 - \delta^2(\beta^2 + 2\beta t + t^2) \\
\Leftrightarrow\ & (\nu^2 + \delta^2)t^2 + 2(\delta\gamma\nu + \beta\delta^2)t = 0 \\
\Leftrightarrow\ & t = 0 \vee t = -2\frac{\beta\delta^2 + \gamma\delta\nu}{\delta^2 + \nu^2} =: t_s.
\end{aligned}$$

For t_s to be a valid solution, we have to check whether (a) t_s is in the domain of M_X, i.e. $t_s \in [-(\alpha + \beta),\ \alpha - \beta]$, (b) t_s indeed solves (6.2) and (c) $t_s < 0$.

As to (a):

$$\begin{aligned} t_s &\le \alpha - \beta \\ \Leftrightarrow -2\delta\nu\gamma - 2\delta^2\beta &\le (\alpha-\beta)(\nu^2+\delta^2) \\ \Leftrightarrow -2\delta\nu\gamma - \delta^2\beta &\le (\alpha-\beta)\nu^2 + \alpha\delta^2 \\ \Leftrightarrow (\alpha-\beta)\nu^2 + 2\delta\nu\gamma + (\alpha+\beta)\delta^2 &\ge 0. \end{aligned}$$

Since $\gamma = \sqrt{\alpha^2-\beta^2} = \sqrt{(\alpha-\beta)(\alpha+\beta)}$ the last expression is equivalent to

$$(\sqrt{(\alpha-\beta)}\nu + \sqrt{(\alpha+\beta)}\delta)^2 \ge 0.$$

In a similar way it can be shown that $t_s \ge -(\alpha+\beta)$.

As to (b): Plugging t_s into (6.2) yields (after some manipulations)

$$M_X(t_s) = \mathrm{e}^{\frac{\delta}{\nu^2+\delta^2}(\Psi - \sqrt{\Psi^2})}$$

with $\Psi = \delta^2\gamma - 2\nu\delta\beta - \nu^2\gamma$. Thus, $M_X(t_s) = 1$ iff $\Psi \ge 0$. Ψ can be considered as a downward open second order polynomial in ν with roots $(\delta/\gamma)(-\beta \pm \alpha)$. Therefore, $\Psi \ge 0$ iff $\nu \le (\delta/\gamma)(\alpha-\beta)$ and $\nu \ge (\delta/\gamma)(-\alpha-\beta)$. Since $\mathbb{E}(X) = \nu + \delta\beta/\gamma$, the latter inequality is fulfilled if X has a positive mean. **Noch die Verbindung zu Homm and Pigorsch (2011) reinbringen.**

As to (c): $t_s < 0$ is equivalent to $\nu > -\delta\beta/\gamma$ (which corresponds to the requirement $\mathbb{E}(X) > 0$).

Summing up, if X is a NIG-distributed random variable with parameters α, β, δ and ν where $0 \le |\beta| < \alpha$, $\delta > 0$ and additionally $\nu \in \left(-\frac{\delta}{\gamma}\beta, \frac{\delta}{\gamma}(\alpha-\beta)\right]$, then the unique strictly positive solution of (6.1) is given by

$$s^* = \frac{1}{2}\frac{\delta^2+\nu^2}{\beta\delta^2+\gamma\delta\nu}.$$

6.3 Moment based representations of the AS index and qualitative results with respect to changes in the first four moments

6.3.1 Moments of a NIG-distributed random variable

The following moments of a NIG-distributed random variable can be derived using the moment generating function in (3.2):

$$\begin{aligned} \text{mean } \mu &= \nu + \frac{\delta\beta}{\gamma}; \\ \text{variance } \sigma^2 &= \frac{\delta\alpha^2}{\gamma^3}; \\ \text{third standardized moment (skewness) } \chi &= 3\frac{\beta}{\alpha\sqrt{\delta\gamma}}; \\ \text{and the fourth standardized moment (excess kurtosis) } \kappa &= 3\frac{\alpha^2 + 4\beta^2}{\delta\alpha^2\gamma}. \end{aligned} \tag{6.3}$$

6.3.2 The inequality $|\chi| < \sqrt{3\kappa/5}$

To see why $|\chi| < \sqrt{3\kappa/5}$ for χ and κ defined in equation (6.3) note that

$$\frac{\chi^2}{\kappa} = 3\frac{\beta^2}{\alpha^2 + 4\beta^2} = 3\frac{(\beta/\alpha)^2}{1 + 4(\beta/\alpha)^2}.$$

Moreover, as $\alpha > |\beta|$ it holds that $(\beta/\alpha) < 1$ and therefore

$$\frac{\chi^2}{\kappa} < \frac{3}{5} \quad \text{and} \quad \chi^2 < \frac{3}{5}\kappa.$$

6.3.3 The AS index in terms of the first four moments of a NIG-distributed random variable

First note that the system of equations (6.3) can (uniquely) be solved for α, β, δ and ν (given that $\chi^2 < (3/5)\kappa$):

$$\alpha = 3\frac{\sqrt{3\kappa - 4\chi^2}}{(3\kappa - 5\chi^2)\,\sigma}; \quad \beta = 3\frac{\chi}{(3\kappa - 5\chi^2)\,\sigma}; \quad \nu = \mu - 3\frac{\chi\sigma}{3\kappa - 4\chi^2}$$
$$\text{and} \quad \delta = \frac{3\sqrt{(3\kappa - 5\chi^2)\,\sigma^2}}{3\kappa - 4\chi^2}.$$

This can be used to rewrite the index of riskiness

$$s^* = \frac{1}{2}\frac{\delta^2 + \nu^2}{\beta\delta^2 + \gamma\delta\nu} = \frac{1}{18}\left(3\kappa\mu - 4\mu\chi^2 - 6\chi\sigma + 9\frac{\sigma^2}{\mu}\right)$$

and the condition for the existence

$$\nu \le \frac{\delta}{\gamma}(\alpha - \beta) \Leftrightarrow \mu \le 3\sqrt{\frac{\sigma^2}{(3\kappa - 4\chi^2)}} \tag{6.4}$$

in terms of these moments.

References

Agarwal, V. and N. Y. Naik (2004). Risks and portfolio decisions involving hedge funds. *Review of Financial Studies 17*, 63–98. [1]

Andersson, J. (2001). On the normal inverse Gaussian stochastic volatility model. *Journal of Business & Economic Statistics 19*, 44–54. [10]

Artzner, P., F. Delbeaen, J. Eber, and D. Heath (1999). Coherent measures of risk. *Mathematical Finance 9*, 203–228. [10]

Aumann, R. and R. Serrano (2008). An economic index of riskiness. *Journal of Political Economy 116*, 810–836. [i, 1, 2, 3, 4, 5, 6, 7, 10]

Barndorff-Nielsen, O. E. (1997). Normal inverse Gaussian distributions and stochastic volatility modelling. *Scandinavian Journal of Statistics 24*, 1–13. [2]

Bernardo, A. E. and O. Ledoit (2000). Gain, loss and asset pricing. *The Journal of Political Economy 108*, 144–172. [8]

Billingsley, P. (1968). *Convergence of probability measures*. John Wiley & Sons. [6, 7]

Bollerslev, T., U. Kretschmer, C. Pigorsch, and G. Tauchen (2009). A discrete-time model for daily S&P 500 returns and realized variations: Jumps and leverage effects. *Journal of Econometrics 150*, 151–166. [10]

Cont, R. (2001). Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance 1*, 223–236. [2]

Dowd, K. (2000). Adjusting for risk: An improved Sharpe ratio. *International Review of Economics & Finance 9*, 209–222. [8]

Eriksson, A., E. Ghysels, and F. Wang (2009). The normal inverse Gaussian distribution and the pricing of derivatives. *The Journal of Derivatives 16*, 23–37. [10]

Goetzmann, W., J. Ingersoll, M. Spiegel, and I. Welch (2007). Portfolio performance manipulation and manipulation-proof performance measures. *Review of Financial Studies 20*, 1503–1546. [1]

Golec, J. and M. Tamarkin (1998). Bettors love skewness, not risk, at the horse track. *Journal of Political Economy 106*, 205–225. [1]

Hansen, L. P. (1982). Large sample properties of generalized method of moments estimators. *Econometrica 50*, 1029–1054. [12, 16]

Harvey, C. R. and A. Siddique (2000). Conditional skewness in asset pricing tests. *Journal of Finance 55*, 1263–1295. [1]

Homm, U. and C. Pigorsch (2011). An economic index of riskiness: A comment. Working Paper, University of Bonn. Available at `http://ect-pigorsch.ect.uni-bonn.de/html/research.html`. [4, 5, 19]

Loistl, O. (1976). The erroneous approximation of expected utility by means of a taylor's series expansion: Analytic and computational results. *aer 66*, 904–910. [13]

Malkiel, B. G. and A. Saha (2005). Hedge funds: Risk and return. *Financial Analysts Journal 61*, 80–88. [1]

Pitts, S. M., R. Grübel, and P. Embrechts (1996). Confidence bounds for the adjustment coefficient. *Advances in Applied Probability 28*, 802–827. [13]

Rydberg, T. H. (2000). Realistic statistical modelling of financial data. *International Statistical Review 68*, 233–258. [2]

Sharpe, W. F. (1966). Mutual fund performance. *Journal of Business 39*, 119–138. [1, 7]

Sharpe, W. F. (1994). The Sharpe ratio. *Journal of Portfolio Management 21*, 49–58. [1]

Sortino, F. A. and L. N. Price (1994). Performance measurement in a downside risk framework. *The Journal of Investing 3*, 59–64. [8]

Tesfatsion, L. (1976). Stochastic dominance and the maximization of expected utility. *The Review of Economic Studies 43*, 301–315. [9]

Treynor, J. L. (1965). How to rate management of investment funds. *Harvard Business Review 43*, 63–75. [7]

Vinod, H. and M. Morey (2001). A double sharpe ratio. In C. F. Lee (Ed.), *Advances in Investment Analysis and Porfolio Mangement*, Volume 8. [14]