Non-Bidding Equilibrium in an Ascending Core-Selecting Auction∗

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Abstract

This paper investigates the perfect Bayesian equilibrium in an ascending-price core-selecting auction, which was recently used in the U.K. spectrum auction. We suppose that there are two goods, two local, and one global bidders. The local bidders demand only one of the goods, whereas the global bidder wants both. Although local bidders generally face the threshold problem and have incentives to underbid, once a bidder becomes a unique remaining local bidder, he bids truthfully. This implies that stopping early induces the remaining bidder to behave truthfully. Either local bidder stops bidding at the beginning in the equilibrium.

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1 Introduction

It is considered important to devise a practical multi-object auction mechanism in the field of market design, particularly in the cases such as spectrum license auctions and

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airport landing slot auctions. Since the Federal Communication Commission’s spectrum auction, the simultaneous ascending auction (SAA) has become a standard format. In the SAA, items are auctioned simultaneously by an ascending-price rule. Every ascending auction is open if a new bid is made for any item, and the auction terminates only when all bidding ceases. Although the SAA is used in many spectrum license auctions across various countries, it does not necessarily perform well in the presence of goods complementarities. For example, if a bidder evaluates several items as complements, he may need to place bids for individual items over their true individual values in order to win everything. Then, he may suffer a loss if he ultimately wins only a part of the complements. Moreover, participants may not bid aggressively to avoid such a loss. This situation is known as the exposure problem.

Package bidding is introduced to solve this problem. If package bids are allowed, the bidder can place a bid for a package of complementary goods. With such bids, he does not face the risk of winning only a part of the complements, and no exposure problem arises. These auctions are called package or combinatorial auctions, and many researchers have studied designs and analyses of package auctions.

The current paper analyzes the equilibrium in an ascending-price package auction, which has recently used for spectrum auctions, under incomplete information. The auction is called the “package (or combinatorial) clock auction,” which is proposed by Porter et al. (2003), Ausubel et al. (2006), and Cramton (2009). In the model of the current study, it is identical to iBundle auction by Parkes and Ungar (2000). We investigate the perfect Bayesian equilibrium and show that the package clock auction results in inefficient outcome and serious low revenue.

In the field of package auction design, a benchmark mechanism is the Vickrey-Clarke-Groves mechanism (the Vickrey auction). It allocates goods efficiently, and bidders follow a dominant strategy of truthful bidding. However, the Vickrey auction is considered impractical owning to certain disadvantages such as low revenue and vulnerability to collusions and shill biddings.1

As an alternative, core-selecting auctions have recently attracted attention. In such auctions, bidders submit bids for packages of goods. The seller selects a total-value-

1For details on these disadvantages of the Vickrey auction, see Ausubel and Milgrom (2006).
maximizing allocation. The payments are determined such that the outcome is in the core with respect to the reported values. Core-selecting auctions can avoid some disadvantages of the Vickrey auction (Day and Milgrom, 2008).

Core-selecting auctions have actually begun to be implemented in certain spectrum auctions; for example, the U.K. telecommunications regulator, Ofcom, has conducted such auctions. This auction comprises two stages: the package clock auction stage followed by a sealed-bid supplementary package auction stage.\(^2\)

Although core-selecting auctions seem attractive and have actually been implemented, they lack incentive compatibility in general. Unfortunately, however, only a few studies have conducted their equilibrium analysis because of its complexity. In most early studies, incentives and equilibrium are analyzed under complete information. A few recent studies advance the analysis of package auctions to the incomplete information case. Goeree and Lien (2009b), Sano (2010a), and Ausubel and Baranov (2010) consider several (sealed-bid) core-selecting auctions in a simple two-good and three-bidder model and examine Bayesian Nash equilibrium.

The current study also investigates a two-good model similar to Goeree and Lien (2009b), Sano (2010a), and Ausubel and Baranov (2010). However, unlike these studies, we examine an ascending-price open-bid auction. It is quite important to consider the ascending-price format since ascending auctions are practically preferred to the sealed-bid format in many cases (Parkes, 2006), and ascending-price formats are actually followed. We show that the difference between sealed-bid and ascending-price formats is large and critical in core-selecting auctions. Moreover, the ascending-price format lowers the performance in the equilibrium.

To compare the preceding studies and the current one, suppose that there are two goods, A and B, and three bidders. Bidders 1 and 2, called local bidders, are interested in the single items A and B respectively.\(^3\) Bidder 3 is a global bidder, who wants both A and B. Suppose that bidders 1 and 2 submit bids \(b_1 = 6\) and \(b_2 = 7\) respectively for their wants and bidder 3 submits a bid \(b_3 = 10\) for the package of AB. Hence, the items are allocated to bidders 1 and 2 in the total-value-maximizing allocation.

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\(^2\)For the design of the U.K. spectrum auction, see Cramton (2009).

\(^3\)The formal model studied in this paper is slightly different. Local bidders evaluate the goods as perfect substitutes.
First, consider the Vickrey auction. In the Vickrey auction, payments are determined such that bidders obtain their marginal contribution to the economy. The total value of the economy without bidder 1 is 10; thus, bidder 1 pays $6 - (13 - 10) = 3$. Similarly, bidder 2 pays $7 - (13 - 10) = 4$. The revenue is $3 + 4 = 7$. This outcome is not in the core because bidder 3 can afford to pay more than 7, and he and the seller block the outcome.

In a core-selecting auction, payments are determined such that any form of blocking never occurs. Payments of bidders 1 and 2, $p_1$ and $p_2$, satisfy $p_1 + p_2 \geq 10$ in any such auction. Particularly, in a “bidder-optimal” core-selecting auction, $p_1 + p_2 = 10$. A notable example of core-selecting auctions is the ascending proxy auction (Ausubel and Milgrom, 2002). In the ascending proxy auction, bidders report their valuations to their proxy agents. Then, the proxy agents participate in a virtual ascending auction with package bidding, given the reported values. In the initial round, proxy agents of bidders 1 and 2 place a bid of $\epsilon$ for their wants. The proxy agent of 3 places a new bid of $3\epsilon$ for package AB in the next round. Then, the proxy agents of 1 and 2 place a new bid of $2\epsilon$ and so on. In each round, every provisionally losing proxy agent raises the bid slightly until the bid reaches the reported value. Thus, in this example, bidders 1 and 2 win with price $p_1 = p_2 = 5$. This outcome is in the bidder-optimal core.

Goeree and Lien (2009b), Sano (2010a), and Ausubel and Baranov (2010) consider several sealed-bid core-selecting auctions in the above 3-bidder case. Sano derives the equilibrium in the ascending proxy auction. Goeree and Lien examine the Bayesian Nash equilibrium in another core-selecting auction called the “nearest-Vickrey auction.” Ausubel and Baranov consider various bidder-optimal core-selecting auctions, including the nearest-Vickrey and the ascending proxy auctions.

The above papers study symmetric equilibrium with respect to local bidders. Their results are summarized as follows. The global bidder has a weakly dominant strategy of truthful reporting. On the other hand, local bidders have an incentive to understate their values. Although local bidders need to cooperate with each other in order to outbid the global bidder, they face a kind of the free-rider problem. In the above example, both local bidders pay 5 in the ascending proxy auction. However, if bidder 1 changes his bid to 4, he wins with a lower price of 4, and bidder 2 pays 6. Thus, each local

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4The nearest-Vickrey auction selects the bidder-optimal core outcome nearest to the Vickrey outcome.
 bidder has an incentive to reduce the bid and make the other local bidder pay more. This situation is known as the threshold problem. Ausubel and Baranov (2010) and the other papers provide the equilibrium comprising symmetric increasing bidding functions. The equilibrium is inefficient and the revenue is lower than that of the Vickrey auction in some cases.

The package clock auction considered in this paper proceeds similar to the ascending proxy auction above; however, it does not use proxy agents. Bidders determine whether to place new bids in each round, and observe the history of the others' behavior. Hence, their strategy can be more complicated.

We show that in the package clock auction, a symmetric increasing equilibrium does not exist under certain conditions, and that perfect Bayesian equilibrium is asymmetric. In a perfect Bayesian equilibrium, one local bidder stops bidding at the beginning of the auction; another local bidder behaves truthfully. The equilibrium is inefficient and results in low revenue. The efficiency and revenue are lower than those of a sealed-bid format. Further, under certain conditions, this is a unique perfect Bayesian equilibrium.

This result strongly depends on the auction dynamics. Similarly to sealed-bid auctions, local bidders initially have an incentive to free-ride on each other and understate. However, if bidder 1 stops bidding first and if bidder 2 observes this, the incentive to understate vanishes because the former can no longer afford to pay. To win the item, bidder 2 needs to outbid the global bidder, so bidder 2 behaves truthfully. From the viewpoint of bidder 1, she can make bidder 2 bid more by stopping before him. Hence, local bidders attempt to cease bidding before each other, thus stopping at the beginning.

We extend the analysis to the case of many goods and bidders. We show that when there are $k$ items, local bidders bid truthfully as long as there are more than $k$ local bidders. After some bidders drop out and the number of active local bidders reduces to $k$, either of them stops immediately by the same consideration. Thus, introducing a new local entrant considerably increases efficiency and revenue.

Contribution of the current paper are as follows. First, we derive a concise equilibrium under incomplete information in the package clock auction which has been actually used. The equilibrium strategy does not directly depend on the value distributions. This is different from the existing analyses on sealed-bid core-selecting auctions. In sealed-
bid formats, the equilibrium strategies crucially depend on the distributions, and we often have no closed-form solutions unless the distribution functions have specific forms.\textsuperscript{5} Moreover, we have a result in the case of many goods and bidders, whereas there is no analysis in the case of many bidders in the sealed-bid formats.

Second, we show that the open-bid format can be inferior to the sealed-bid one in core-selecting auctions. The free-rider problem is more serious in the open-bid format than in the sealed-bid one. This is contrary to the perception that ascending auctions are often preferred to sealed-bid ones in both single- and multi-object auctions (Milgrom and Weber, 1982; Parkes, 2006; Compte and Jehiel, 2007).

1.1 Related Literature


With regard to specific pricing rules in core-selecting auctions, Day and Cramton (2008) propose the nearest-Vickrey auction; which selects an outcome in the bidder-optimal core nearest to the Vickrey outcome. Erdil and Klemperer (2010) propose the "reference rule," which selects an outcome in the bidder-optimal core as independently from the winning bids as possible.

As we have already mentioned, Goeree and Lien (2009b), Sano (2010a), and Ausubel and Baranov (2010) analyze the Bayesian Nash equilibrium of core-selecting auctions. Goeree and Lien and Sano examine the equilibrium in the independent-value case. Goeree and Lien also show that there is no Bayesian-incentive-compatible core-selecting auction in general. Sano (2010a; 2010b) provides a necessary and sufficient condition for truthful bidding when each bidder is interested in a particular bundle of goods. Ausubel and Baranov consider the case of correlated values and claim that core-selecting auctions

\textsuperscript{5}See Ausubel and Baranov (2010) for example.
perform well in the presence of correlation.

The package clock auction studied in the current paper is proposed by Porter et al. (2003) through laboratory experiments. They claim that this auction performs well and achieves high efficiency. Related experiments are also conducted by Brunner et al. (2010) and Kagel et al. (2010).

The current paper is also related to the analyses of the SAA. Particularly, some studies report the low revenue equilibria of the SAA in models similar to ours. Such a low revenue equilibrium can arise from the exposure problem, demand reduction, or collusion. The exposure problem is studied by Goeree and Lien (2009a) in an incomplete-information model comprising local and global bidders. Demand reduction is studied mainly in multi-unit uniform-price auctions (Engelbrecht-Wiggans and Kahn, 1998; Ausubel and Cramton, 2002). In the open-bid format, Ausubel and Schwartz (1999) show that the low revenue outcome is a unique subgame perfect equilibrium under complete information. Brusco and Lopomo (2002) and Engelbrecht-Wiggans and Kahn (2005) examine collusions in the SAA and show the existence of a collusive perfect Bayesian equilibrium that achieves low revenue. The low revenue equilibrium studied in the current paper originates from the free-rider problem. As we discuss in detail later, the logic behind the low revenue is unique.

The remainder of the paper proceeds as follows. In section 2, we formulate the model and the auction. In section 3, we analyze the equilibrium. We show that there exists a perfect Bayesian equilibrium in which local bidders want to stop bidding at the initial period of the auction. Under certain conditions, such an equilibrium is a unique one comprising nondecreasing strategies. In section 4, we extend the analysis. We provide an equilibrium in the case of many goods and bidders. We show that in the case of many local bidders, they behave truthfully at first until some bidders drop out.

2 The Model

A seller allocates two items, A and B. There are three bidders: \( N \equiv \{1, 2, 3\} \). All bidders have quasi-linear utilities and are risk-neutral. Bidders 1 and 2 are local bidders and are interested in only one of the goods. They want either A or B, and they evaluate equally. Bidder 3, on the other hand, is a global bidder, who wants both items. He assigns a value
of 0 to each single item. Let \( u_i : \{\emptyset, A, B, AB\} \to \mathbb{R}_+ \) be bidder \( i \)'s utility function, and normalize \( u_i(\emptyset) = 0 \) for all \( i \). For each local bidder, \( u_i(A) = u_i(B) = u_i(AB) = v_i \). For each global bidder, \( u_i(A) = u_i(B) = 0 \) and \( u_i(AB) = w_i \). Henceforth, we use a female pronoun for bidder 1 and a male pronoun for bidders 2 and 3.

Assume that all the situation provided above is publicly known. However, the values are private information. Each local value \( v_i \) is local bidder \( i \)'s private information, which is drawn from a distribution \( F \) on interval \( V = [0, 1] \). Similarly, global value \( w_3 \) is global bidder 3’s private information, which is drawn from a distribution \( G \) on interval \( W = [0, a] \), where \( a \geq 2 \). Let \( F \) have density \( f \) and \( f(v) > 0 \) for all \( v \in (0, 1) \) and \( G \) have density \( g \) and \( g(w) > 0 \) for all \( w \in (0, a) \).

2.1 The Auction

Let \( r \in [0, 1) \) be the reserve price of each item. We assume that the reserve prices of the items are the same and that the reserve price for the package is \( 2r \). Before the auction starts, the bidders decide whether to participate in it. Suppose that the decisions about participation are publicly observed. We consider the package (combinatorial) clock auction proposed by Porter et al. (2003), which proceeds as follows. There are two price clocks \( (p_A, p_B) \). The price for the package of the items is always set to the sum of the prices of the single items: \( p_{AB} = p_A + p_B \). The auction starts at initial price vector \( (r, r) \).

Each bidder can place bids for either A or B individually or package AB. Each price clock increases continuously and simultaneously at a constant speed of 1 as long as more than one bidder places bids for the item. At each time \( t \in [0, T] \),\(^6\) bidders decide whether to continue bidding. Bidders are restricted by an activity rule: once a bidder stops bidding, he/she can no longer place a new bid. The price of an item stops increasing when the aggregate demand for that item becomes 1 or 0. The price clocks never decrease, and the auction terminates if all the prices stabilize. The seller allocates the items efficiently; taking all bids during the auction into consideration.

For simplicity, assume that bidders 1 and 2 submit bids for single items A and B.

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\(^6\)To ensure that the auction ends, suppose that it necessarily ends at a sufficiently large number \( T > a \). If the auction does not terminate at \( T \), suppose that items are allocated randomly among continuing bidders with the prices at \( T \).
respectively. In addition, assume that the global bidder submits bids for package AB. The auction outcome in the case that all bidders participate is summarized as follows:

Case 1: Bidder 3 stops first at $p_A = p_B = \frac{1}{2}p_3$. Then, bidders 1 and 2 win the single items A and B respectively. Both bidders 1 and 2 pay $p_A = p_B = \frac{1}{2}p_3$.

Case 2: Bidder 1 stops first at $p_A = p_1$, and bidder 3 stops next at $p_3 = p_1 + p_B$. Then, bidders 1 and 2 win A and B respectively. Bidder 1 pays $p_A = p_1$, and bidder 2, $p_B = p_3 - p_1$.

Case 3: Bidder 2 stops first at $p_B = p_2$, and bidder 3 stops next at $p_3 = p_A + p_2$. Then, bidders 1 and 2 win A and B respectively. Bidder 1 pays $p_A = p_3 - p_2$, and bidder 2, $p_B = p_2$.

Case 4: Bidders 1 and 2 stop first and second at $p_A = p_1$ and $p_B = p_2$ respectively. Then, bidder 3 wins both items with price $p_{AB} = p_1 + p_2$.\footnote{If bidder 3 does not participate, the items are allocated to bidders 1 and 2 with $r$, provided they participate. If both local bidders do not participate, the items are allocated to bidder 3 with $2r$, provided he participates. If either bidder 1 or 2 does not participate, the auction is similar to a single-item English auction.}

Note that local bidders do not switch the bid items even if $p_A = p_B$. However, even if local bidders switch their bids, the auction outcome should remain the same. See Appendix A for the bid switching. Another assumption is imposed for the convenience of examining the equilibrium as follows.

**Assumption 1** If two or more bidders stop simultaneously, then only one stopper is selected randomly and the others are rejected.

In addition, assume that if a bidder stops, the prices temporarily stabilize for a second. Even if a stopping is rejected, the bidders can make a choice of stopping during this interval at the same prices again. Assumption 1 implies that ties are broken randomly. In addition, note that Assumption 1 is applied in the case that bidders 1 and 2 simultaneously stop. We can dispense with Assumption 1 in such a case if the price increase is discrete. See Appendix B for the discrete case.
With ignoring the reserve prices, the auction is identical to Ausubel and Milgrom’s (2002) ascending package auction without proxy bidding and also to Parkes and Ungar’s (2000) iBundle auction. Hence, it is a core-selecting auction. If bidders simultaneously report their ceiling prices in advance and if proxy agents automatically raise bids until the reported value, we obtain the ascending proxy auction. The difference between the proxy auction and ours is the use of proxy agents. In open-bid model, bidders can change their stopping price contingent on the observed history of the auction. The difference between this auction and the SAA is package bidding. In the SAA, the global bidder must place bids for both individual items. Hence, if bidder 1 stops first at $p_1$, then bidder 3 must buy item A with $p_1$ regardless of the result of item $B$.\footnote{If bidder 2 can switch the bid items, the situation changes slightly. In such a case, arbitrage occurs and both prices continue increasing simultaneously. Nevertheless, bidder 3 will need to buy either A or B.}

2.2 Strategy and Equilibrium

In the auction, bidders decide whether to stop at every moment $t \in [0,T]$. A bidder’s strategy is a mapping from his/her value and all possible histories to a binary choice set \{continue, stop\}. We consider a \textit{reduced strategy} where a bidder determines when he/she stops, taking the other bidders’ dropout information into consideration.

We limit our attention to pure strategy. Let $\beta_i : V \ (or \ W) \times H \rightarrow \mathbb{R}_+$ be bidder $i$’s strategy, where $H$ denotes the set of histories. A history $h \in H$ comprises observed stoppings of bidders: $h = \{(i, p_i), \ldots\}$. For example, $h = \{(2, p_2)\}$ indicates that bidder 2 stopped at $p_B = p_2$. Given observed stoppings $h$, $\beta_i(v_i \ (or \ w_i), h)$ decides $i$’s stopping price. If no one has stopped by then, let $h \equiv \emptyset$.

The equilibrium concept is perfect Bayesian equilibrium.\footnote{To be precise, we focus on the equilibrium such that bidders play undominated strategies. Hence, the equilibrium is undominated perfect Bayesian equilibrium.} For simplicity, we refer to it as “equilibrium.”
3 Main Results

If a bidder does not participate, the auction is either reduced to a single-unit English auction or cancelled. Hence, we limit our attention to the case where all bidders participate. We first observe that bidder 3 has a weakly dominant strategy of truthful bidding. He wins the auction if and only if he continues to bid until both bidders 1 and 2 stop; that is, the sum of the prices reaches \( p_1 + p_2 \). In addition, the amount of the payment is determined by the bids of the local bidders \((p_1 + p_2)\). This is the same as a familiar single-item English auction. Hence, truthful bidding is clearly a weakly dominant strategy for bidder 3: \( \beta_3(w_3, h) = w_3 \) for all \( h \).

Similarly, local bidders 1 and 2 also have a weakly dominant strategy after the subgame in which alternate bidders stop bidding. Suppose that bidder 2 stops bidding at price \( p_2 \). Then, the price of item B stabilizes, and the auction is reduced to a simple English auction of item A. Hence, it becomes a weakly dominant strategy for bidder 1 to bid truthfully after this subgame. Formally, bidder 1 wins the auction if and only if she continues bidding until \( p_A = p_3 - p_2 \). In addition, she pays amount \( p_3 - p_2 \) when she wins. Bidder 1 plays a single-unit ascending auction against bidder 3, who is restricted by a fixed amount \( p_2 \). Hence, regardless of the price that bidder 2 stops at, it is a dominant strategy for bidder 1 to bid up to her true value after she observes bidder 2's stopping: \( \beta_i(v_i, (j, p_j)) = v_i \) for \((i, j) = (1, 2), (2, 1)\).\(^{10}\) Hereafter, we focus on the equilibrium in which bidders choose these dominant strategies.

3.1 The Extremely Free-Riding Equilibrium

For simplicity of notation, \( \beta_i(v_i, \emptyset) < r \) represents the strategy such that \( i \) does not participate in the auction.\(^{11}\) Theorem 1 claims that there exists an equilibrium where both local bidders choose to stop at the initial price under a certain condition.

Theorem 1 There exists a perfect Bayesian equilibrium in which \( \beta_i(v_i, \emptyset) = \min\{v_i, r\} \)

\(^{10}\)Precisely, if \( p_2 > v_1 \), bidder 1's optimal strategy is to drop out as soon as possible. However, we ignore this since continuing bids over the true value is obviously not optimal.

\(^{11}\)For the global bidder, \( \beta_i(w_i, \emptyset) < 2r \).
for each $i = 1, 2$ and all $v_i \in V$ if and only if
\[
\int_r^1 \frac{1}{1 - F(r)} \left( G(x + r) - G(2r) \right) f(x) - 1 \, dx \geq 0. \tag{1}
\]

**Proof.** We suppose that bidders follow the weakly dominant strategies of truthful reporting: $\beta_i(v_i, \{(j, p_j)\}) = v_i$ for each $(i, j) = (1, 2), (2, 1)$, all $v_i \in V$, and all $p_j$. In addition, $\beta_3(w_3, h) = w_3$ for all $h$ and $w_3 \in W$ regardless of the reserve price. It is also obvious that local bidders never participate if $v_i < r$. Further, non-participation is dominated by the truthfully bidding strategy when $v_i \geq r$. Hence, local bidders participate if and only if $v_i \geq r$.

Suppose that bidder 2 stops immediately at $r$ regardless of his realization of $v_2 (\geq r)$. If bidder 1 stops at $r$ and if her stopping is accepted, she free-rides on bidder 2 and her interim expected payoff is $(v_1 - r) \Pr\{v_2 + r > w_3 | v_2 \geq r, w_3 \geq 2r\}$. On the other hand, if 1’s stopping is rejected, or if she chooses any other strategy, then she consequently bids up truthfully. Therefore, the interim expected payoff is given by $\int_{2r}^{v_1 + r} (v_1 + r - w_3) dG(w_3 | w_3 \geq 2r)$. Hence, stopping at $r$ is optimal if
\[
(v_1 - r) \Pr\{v_2 + r > w_3 | v_2 \geq r, w_3 \geq 2r\} \geq \int_{2r}^{v_1 + r} (v_1 + r - w_3) dG(w_3 | w_3 \geq 2r) \tag{2}
\]
for all $v_1 \geq r$.

The left and right hand sides of (2) is linear and convex functions of $v_1$ respectively. Since (2) holds with $v_1 = r$, it is satisfied for all $v_1 > r$ if it holds only with $v_1 = 1$:
\[
\frac{1 - r}{1 - F(r)} \int_r^1 (v_2 + r) dG(w_3) dF(v_2) \geq \int_{2r}^{1+r} (1 + r - w_3) dG(w_3). \tag{3}
\]
(3) yields (1) with some calculation. ■

Note that (1) holds when $F$ is the uniform distribution. Roughly speaking, (1) is a condition on the distribution of local values. It requires that the distribution of local values, $F$, yields a high value. Particularly, when there is no reserve price, $r = 0$, the above equilibrium exists if $F$ first-order-stochastic-dominates the uniform distribution.

**Corollary 1** Suppose $r = 0$. The perfect Bayesian equilibrium described in Theorem 1 exists if $F$ first-order-stochastic-dominates the uniform distribution.
**Proof.** Suppose \( r = 0 \). Then (1) yields
\[
\int_0^1 G(x)(f(x) - 1)dx \geq 0.
\]

(4) is equivalent to
\[
\int_0^1 g(x)(F(x) - x)dx \leq 0.
\]

The equilibrium strategy is symmetric with respect to local bidders, however, the equilibrium behavior is asymmetric. Either one of 2 local bidders stops at the initial price, and another local bidder behave truthfully. Although Assumption 1 seems crucial in the equilibrium, it is not. See Appendix B.

The open-bid format of the auction is critical to Theorem 1. Even if local bidders like to free-ride on each other, once either one of them stops bidding, the other local bidder must change his/her behavior to truthful one. Early stopping plays a role of a commitment to not paying more than the current price. Hence, the activity rule is also critical to the result.

The asymmetric equilibrium, in which one local bidder makes no bid and another behaves truthfully, can exist in the ascending proxy auction. However, it is far less likely to exist than in the package clock auction. This is because non-bidding leads to low winning probability and is unprofitable, especially when the bidder has a high value. In our clock auction, however, non-bidding is sustainable since the “truth-telling bidder” always seeks to preempt stopping. Once the “stopping bidder” deviates and raises the bid, another bidder immediately takes the place of the stopping bidder.

Another remark on Theorem 1 is the robustness to value distributions. In the sealed-bid formats such as the ascending proxy auction, the equilibrium strategy strongly depends on value distributions \( F \) and \( G \). However, the equilibrium strategy studied here does not directly depend on the distributions. In addition, Ausubel and Baranov (2010) report that correlation between local values improves the performance of the ascending proxy auction. Although we do not analyze the case of correlation, however, Theorem 1 will hold even if there is correlation between local values to some extent.

\[\text{See Ausubel and Baranov (2010).}\]
3.2 Uniqueness

Extreme free-riding is a unique outcome of the auction in the sense that $\beta_i(v_i, \emptyset) = r$ for $i = 1$ or $2$ is necessary for equilibrium under other conditions. The construction of Theorem 1 is straightforward, and one might wonder whether there exists another reasonable equilibrium in which both local bidders place a certain amount of bids. However, such an equilibrium does not exist if we limit our attention to nondecreasing strategies and if the distribution of the global value satisfies a certain condition.

In Theorem 2, we focus on the case where each bidder’s strategy is nondecreasing. The standard single-object auction rules ensure that the probability of winning increases with bidding. This implies that the bidders’ payoffs satisfy the single crossing property and that any equilibrium strategy should be nondecreasing. Hence, we need not consider a decreasing equilibrium. However, given the sequential rationality, $\beta_i(v_i, h) = v_i$ for $h \neq \emptyset$, each bidder’s associated expected payoff\(^\text{13}\) $\pi_i(b_i, v_i)$ may not satisfy the single crossing property (i.e., $\frac{\partial^2 \pi_i}{\partial b_i \partial v_i} < 0$ in some cases). Since raising bids may decrease the probability of winning the item, $\beta_i(\cdot, \emptyset)$ may be decreasing in an equilibrium. Nevertheless, it seems appropriate to focus on the case of a nondecreasing equilibrium.

**Theorem 2** Suppose $g$ is nonincreasing in interval $(r, 2)$. If each $\beta_i$ is nondecreasing in $v_i$ and if a strategy profile, $(\beta_1, \beta_2, \beta_3)$, constitutes a perfect Bayesian equilibrium, then $\beta_i(v_i, \emptyset) = r$ for some $i \in \{1, 2\}$ and all $v_i \in [r, 1)$.

**Proof.** See Appendix C.

Theorem 2 shows that in the package clock auction, a symmetric increasing equilibrium does not exist under the condition. Suppose that bidder 2 chooses a continuous increasing strategy in which $\beta_2(\hat{v}, \emptyset) = p_2 > r$. Consider whether the symmetric strategy $\beta_1(\hat{v}, \emptyset) = p_2$ can be an equilibrium. Suppose that $v_1 = \hat{v}$ and that the prices approach $p_2$. Note that if bidder 2 stops first, then bidder 1 has to raise the bids up to $\hat{v}$. Then, she pays the amount $p_2 \leq p \leq \hat{v}$ when she wins. On the other hand, if bidder 1 stops immediately, then bidder 2 bids up to the true value conversely. Hence, bidder 1 can compel bidder 2 to pay $p - p_2$ by slightly lowering the dropout price from $p_2$. Moreover,

\(^{13}\)A bidder’s associated expected payoff $\pi_i(b_i, v_i) \equiv E[u_i(x_i((b_i, v_i), \beta_{-i})) - p_i((b_i, v_i), \beta_{-i})|v_i]$, where $x_i$ denotes the allocation for $i$ and $\beta_i(v_i, \emptyset) = b_i$.\(^{14}\)
at this moment, bidder 1 expects bidder 2’s value to be $v_2 \geq \hat{v} - \epsilon$, while her own value is $\hat{v}$. Although stopping earlier decreases the probability of winning, this cost is relatively small with the assumption on $g$. Therefore, it is more profitable to stop immediately than to continue until $p_2$.

This consideration implies that for any $v \in (r, 1)$ and any $\beta_2(v, \emptyset) > r$, bidder 1’s best reply is $R_1(v) < \beta_2(v, \emptyset)$, and vice versa. That is, each local bidder wants to stop first. Consequently, each bidder’s stopping price in the equilibrium falls to $r$.

In contrast to (1), the assumption on the distribution of the global value is necessary; the winning probability of local bidders should not increase excessively by increasing local bids. When this condition is violated, the marginal profit of winning the good is so large that local bidders have a lower incentive to free-ride. Then, there may exist an equilibrium where local bidders behave more aggressively.

Notably, Theorem 2 does not assure that there exists an equilibrium such that $\beta_i(v_i, \emptyset) = r$ when $g$ is nonincreasing. A pure strategy equilibrium may not exist when (1) does not hold. The inequality (1) is violated when $F$ frequently yields a low value. In such a case, a local bidder with a high value prefers bidding truthfully to free-riding since his or her counterpart tends to have a low value. Then, those with high values want to continue bidding at the beginning of the auction. However, such a strategy cannot constitute an equilibrium by Theorem 2.

### 3.3 Efficiency and Revenue

The package clock auction has serious inefficiency and low revenue. To observe this, we compare the efficiency and revenues of the ascending proxy auction and the Vickrey auction. Sano (2010a) and Ausubel and Baranov (2010) derive the symmetric Bayesian Nash equilibrium of the ascending proxy auction. Ausubel and Baranov (2010) also report the efficiency and revenue of the ascending proxy auction in the case of uniform distributions.$^{14}$

Suppose that $r = 0$ and that all values are drawn from uniform distributions. In a Bayesian Nash equilibrium of the ascending proxy auction, each local bidder follows the

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$^{14}$The global value $w_3$ is distributed on $[0, 2]$. 

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strategy

\[ \beta_i(v_i) = \max\{0, 1 + \log v_i\}. \]

The ascending proxy auction achieves approximately 87% efficiency. However, the package clock auction achieves only 75% efficiency.

The expected revenue of the ascending proxy auction is 0.536, whereas that of the package clock auction is 0.417. Notice that the revenue of the Vickrey auction is 0.583; thus, the package clock auction is inferior to both the ascending proxy auction and the Vickrey auction.

The equivalence between ascending and sealed-bid auctions does not hold. Our results provide a remark on the existing notion that dynamic auctions are preferable to static ones in many cases. In the package auctions problem, it is generally difficult for bidders to report values for many possible packages of goods in sealed-bid auctions. On the other hand, ascending auctions are easy to operate since bidders only determine an optimal bundle of goods according to a current price vector (Cramton, 2009; Parkes, 2006). Moreover, ascending auctions reveal price information to bidders and are considered to promote competition. However, our results show that the package clock auction promotes bidders’ preemptive stopping because of history-dependent strategies.

3.4 Discussion

3.4.1 Sealed-Bid Stage After Clock Auction

The extremely free-riding outcome stems from the ascending-price open-bid format and the activity rule. Once a local bidder stops bidding, he/she cannot place a higher bid any longer. Early stopping plays a role of a commitment, and another local bidder must modify the bidding behavior. Therefore, if the maximum bid is restricted by the stopping price, the results do not change even when there is a supplementary sealed-bid stage after the clock auction.

The recent U.K. spectrum auction has adopted a hybrid auction of the clock auction studied thus far and a supplementary sealed-bid auction. The package clock auction is conducted at the first stage. Then, a sealed-bid core-selecting auction follows at the second stage. A bidder-optimal core-selecting auction is used for the second stage.
To make the first stage relevant, an activity rule is imposed on bidders. A natural activity rule is such that if a bidder stops at \((\hat{p}_A, \hat{p}_B)\) in the clock auction, his/her maximum possible bids at the second stage must be \((\hat{p}_A, \hat{p}_B, \hat{p}_A + \hat{p}_B)\) for each single item and the package. Under this activity rule, the result does not change even with the second stage. If bidder 1 stops at the reserve price \(r\), she cannot place any bid more than \(r\) in the second stage. Hence, the situation does not change for the other bidders, and the same equilibrium exists.

The analysis will be different if bidders are allowed to place higher bids than the stopping prices. The commitment effect of stopping becomes weak, so that the equilibrium may become similar to that in a sealed-bid auction.

### 3.4.2 Low Revenue in the SAA

A low revenue equilibrium exists in the package clock auction. Similarly, several low revenue problems in the SAA have been observed in the existing literature. However, the low revenue studied in the current paper is completely different from that in the existing results on the SAA.

The low revenue equilibrium of the SAA can arise from the exposure problem, demand reduction, or collusion. The exposure problem is studied by Goeree and Lien (2009a). They consider a model comprising local and global bidders. In the SAA, local bidders follow a dominant strategy of truthful bidding. However, because of the exposure problem, global bidders face the risk of negative profits and drop out early in the auction. In addition, they indicate an inverse relationship between the number of bidders and the duration of bidding by the global bidder, since the anticipated winning price of each item increases. These properties do not necessarily depend on the ascending-price format. A similar result is found in simultaneous sealed-bid auctions (Krishna and Rosenthal, 1996).

Demand reduction arises in multi-unit uniform-price auctions with multi-unit demands (Ausubel and Cramton, 2002). In a uniform-price auction, bids for the second or subsequent units can affect the final price. Since bidders have pricing power to some extent, they have incentives to reduce their demand and lower the price. Engelbrecht-Wiggans and Kahn (1998) show that there exists a zero-revenue equilibrium under certain
conditions. A similar incentive arises in the SAA when goods are substitutes (Ausubel and Schwartz, 1999). The free-rider problem studied in this paper is similar to demand reduction in the sense that local bids have some pricing power because of core-selecting property.

A collusive perfect Bayesian equilibrium is considered by Brusco and Lopomo (2002) and Engelbrecht-Wiggans and Kalm (2005). These papers show that there exists a collusive low revenue equilibrium in the SAA using the following strategy. Suppose there are two goods A and B and two bidders, who have an additive valuation. In the initial period of an SAA, bidders 1 and 2 place a bid for only A and B respectively. If each bidder actually does so, the auction ends. If a bidder deviates and places any other bid, then both bidders adopt a competitive (truthfully bidding) strategy.

This collusive equilibrium strongly depends on the open-bid format of the SAA, regardless of whether package bidding is possible. A similar equilibrium will exist even when package bidding is allowed. However, this logic of low revenue is different from that of the current paper. In our model, each bidder is interested in only one type of goods. It is impossible to coordinate and split goods among bidders in advance. Local bidders neither cooperate nor collude. Rather, they compete with each other to preempt stopping and free-riding.

4 Many Goods and Bidders

In this section, we extend our analysis to the case where there are many goods and many local and global bidders. Our results are independent of the number of bidders. Suppose that there are $k$ goods; $m$ local bidders; and $n$ global bidders. Local bidders want only one of the goods, and they evaluate the goods equally as perfect substitutes. On the other hand, global bidders want all the goods, and they evaluate 0 for any part of the goods. Suppose that global values $w_i$’s are drawn from a distribution $G$ on $[0,k]$.

In the auction, local bidders make bids for the item with the lowest price.\footnote{Local bidders can switch items.} By arbitrage, all the individual prices are the same when at least two local bidders continue bidding. Theorem 3—an extension of Theorem 1—proves the existence of a free-riding
equilibrium even in the presence of many goods and bidders, under certain conditions.

**Theorem 3** Suppose that there are \( k \) goods, \( m \) local and \( n \) global bidders. If

\[
\int_s^1 \left( G(x + (k - 1)s) - G(ks) \right) \left( \frac{1 - s}{1 - F(s)} f(x) - 1 \right) dx \geq 0
\]

for each \( l = 1, \ldots, n \) and all \( s \in [r, 1) \), then the following strategies constitute a perfect Bayesian equilibrium:

1. Each global bidder follows a truthful strategy.
2. If \( m' \geq k + 1 \) local bidders continue bidding, then each local bidder follows a truthful strategy.
3. If \( 2 \leq m' \leq k \) local bidders continue bidding (and if a global bidder continues bidding), then each local bidder stops immediately.
4. If a bidder is a unique continuing local bidder, then he or she follows a truthful strategy.

In particular, the above equilibrium exists when \( F \) is the uniform distribution.

**Proof.** See Appendix C.

Factors relating to the global bidders remain unchanged; they retain the dominant strategy property. Local bidders, however, first follow the truthful strategy until only \( k \) of them remain, which results in preemptive stoppings.\(^{16}\)

The free-rider problem becomes disastrous as the number of the goods increases and \( m = k \). In such a case, only one of \( k \) local bidders behaves truthfully and the others bid 0. Hence, the serious inefficiency arises.

On the other hand, note that the local bidders behave truthfully as long as more than \( k \) of them are active in the auction. To be qualified to free-ride, each local bidder has to be at least the \( k \)th highest of all the local bidders. Hence, they purely compete with each other regardless of global bidders, and the threshold problem does not arise. This implies that efficiency and revenue are considerably increased by promoting new local entries in the auction.

\(^{16}\)A similar idea appears in Brusco and Lopomo (2002). They consider collusions in the two-good SAA. When there are many bidders, they behave truthfully until two bidders remain. Then, the remaining bidders split the goods, and the auction ends.
5 Conclusion

We examine the equilibrium of the package clock auction, which was recently used for the U.K. spectrum auction. Bidders can choose history-dependent strategies in dynamic auctions. Since local bidders can make each other bid aggressively and truthfully by stopping early, each of them wants to stop bidding first. As a result, each local bidder attempts to stop at the beginning of the auction regardless of valuations. In the equilibrium, either one of local bidders actually stops at the initial round of the auction, and the other continues bidding truthfully. The equilibrium outcome is unique under certain conditions. This result is different from the sealed-bid package auctions studied by Ausubel and Baranov (2010), Goeree and Lien (2009b), and Sano (2010a).

The equilibrium suffers a serious low revenue problem, which differs from that of the SAA. The latter is caused by the exposure problem, demand reduction, or collusion. In the present model, these problems do not exist; however, the free-rider problem lowers revenue. This problem exists even in sealed-bid package auction. However, it is aggravated by the ascending-price format, and it leads to preemptive stopping by the local bidders. This is contrary to the perception that ascending auctions are often preferred to sealed-bid ones in practice.

If there are many local bidders, they bid truthfully at first. This implies that efficiency and revenue are considerably improved by promoting new local entries in the auction. It is a notable suggestion for practical auction design.

A Bid Switching by Local Bidders

We suppose that when a bidder is a unique continuing local bidder, he places bids for only one of the goods even if \( p_A \neq p_B \). It is justified even if we allow bidders switching the bid items. This is because the auction outcome is determined from all the bids placed throughout the auction. The justification can be done in the case of both the original design by Porter et al. (2003) and another design used for the U.K. spectrum auction. In the former design, the outcome is determined only through the ascending price auction. On the other hand, the latter has a supplementary sealed-bid stage after the ascending auction.
A.1 The Auction without Supplementary Stage

We first consider the case of no supplementary stage, which is considered by Porter et al. (2003). Suppose that switches are allowed. Suppose that bidder 2 stops first at \( p_A = p_B = p_2 \), and that bidders 1 and 3 are left. Since bidder 1 will switch the bids, the prices continue to rise with \( p_A = p_B \). Suppose that bidder 3 then stops at \( p_A = p_B = \frac{1}{2}p_3 > p_2 \). Note that the final allocation has not been determined yet. The bid of bidder 3 for the package is \( p_{AB} = p_3 \). On the other hand, if local bidders obtain the goods, the total reported value is \( z \geq \frac{1}{2}p_3 + p_2 \). To identify the total-bid-maximizing allocation, the auctioneer should raise the prices further until \( p_3 - p_2 \). Thus, bidder 1 has to raise her bids until the sum of local bids exceeds the package bid regardless of switching.

Bidder 1 can alternatively place bids only for B and outbid bidder 2. However, such a strategy is so costly that bidder 1 never plays it actually and that it never constitutes an equilibrium. Hence, such a strategy cannot be a threat to bidder 2 either.\(^{17}\)

For global bidders, we suppose that if bidder 3 wins, his payment is determined as the sum of bids by local bidders, not as the final package price. This is because the final package price may exceed bidder 3’s bid. It is notable that the auction rule is identical to Ausubel and Milgrom’s (2002) ascending auction without proxy agents and Parkes and Ungar’s (2000) iBundle auction.

A.2 The Auction with Supplementary Stage

The U.K. spectrum auction adopts a different design. The ascending auction terminates if no excess demand exists. There is a supplementary stage of sealed-bid auction after the ascending auction. Bidders submit sealed bids for any package of goods. In the supplementary stage, bidders are restricted by the activity rule: they cannot submit a higher bid than the stopping price in the ascending auction. The allocation and the payments are determined by all the bids in both ascending and sealed-bid auctions. They are determined such that the outcome is in the bidder-optimal core.

Suppose that switches are allowed. Suppose that bidder 2 stops first at \( p_A = p_B = p_2 \),\(^{21}\)

\(^{17}\)When bidder 2 stops at \( r \), this strategy is in fact not costly, and it prevents bidder 2 from free-riding. However, we ignore this since it is a limit case.
and that bidder 3 then stops at $p_A = p_B = \frac{1}{2}p_3 > p_2$ because of bid switching. Then, the ascending auction terminates. In the following supplementary stage, bidders 2 and 3 are restricted by the activity rule, so that they cannot raise the bids further. However, bidder 1 can submit any bids for individual goods.

Since bidders 2 and 3 cannot change their bids, the supplementary stage is simply a single-person decision problem of bidder 1. By Day and Milgrom (2008), submitting $\min\{v_1, p_3 - p_2\}$ is a best response in every core-selecting auction. Hence, when bidder 1 wins, her payment must be $p_3 - p_2$, which is the same as the rule described in section 2.1.

For global bidders, when bidder 3 wins, his payment is determined as the sum of bids by local bidders, since the outcome is selected from the bidder-optimal core. Thus, our simplification is justified by the supplementary stage and perfection.

**B Discrete Price Increase**

In the case of multiple efficient allocations, Assumption 1 is natural. However, the case that bidders 1 and 2 stop simultaneously appears to be slightly unusual. If we consider a discrete clock auction instead of the continuous one, we need not impose this assumption for such a case.

Suppose that we modify the rule of the auction to a multi-round auction as follows. At round $t = 0$, the initial prices are set as $(p_A^0, p_B^0) = (r, r)$. Bidders decide whether to participate. At round $t = 1, 2, \ldots$, the price of item $k$ is raised by $\epsilon$ if there is excess demand for $k$: $p_k^t = p_k^{t-1} + \epsilon$. Otherwise, $p_k^t = p_k^{t-1}$. Each bidder makes a choice from \{continue, stop\}. Suppose that Assumption 1 is applied only in the case of tie-breaking. If bidders 1 and 2 simultaneously stop, then the auction ends and bidder 3 wins. Further, suppose that price increment $\epsilon > 0$ is negligible.

Bidder 3 follows a truthful dominant strategy. In addition, $\beta_i(v_i, h) = v_i$ for $h \neq \emptyset$ also holds for local bidders. The following proposition is a corresponding result of Theorem 1.

**Proposition 1** Associated with the dominant strategy properties, the following strategies

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18We do not need $p_A = p_B$ when bidder 3 stops.
constitutes a perfect Bayesian equilibrium if and only if (1) holds:

1. Bidder 1 stops at \( t = 1 \) if \( v_1 \geq r \).

2. Bidder 2 stops at \( t = 2 \) if \( v_2 > r \) and if bidder 1 chooses to “continue” at \( t = 1 \).

Clearly, there exists another equilibrium such that bidders 1 and 2 are inverted.

C Proofs

C.1 Proof of Theorem 2

After the observation of participations, bidders know that each bidder’s value is distributed on \( [r, 1] \) or \( [2r, a] \). Hence, let \( r = 0 \) without loss of generality.

Suppose that bidders follow the dominant strategies of truthful reporting: \( \beta_i(v_i, h) = v_i \) for \( h \neq \emptyset \), each \( i = 1, 2 \), and all \( v_i \in V \), and \( \beta_2(w_3, h) = w_3 \) for all \( w_3 \in W \) and all \( h \). Then, the auction game is reduced to a static two-player game of bidders 1 and 2. We represent \( \beta_i(\cdot, \emptyset) \) simply as \( \beta_i(\cdot) \).

Let \( R_i : V \to [0, T] \) be bidder \( i \)’s best response function, implicitly given bidder \( j \)’s strategy \( \beta_j \). Obviously \( \beta_i(v) \leq v \) for each \( i = 1, 2 \) and all \( v \). Proof is constructed by using following 3 lemmas.

**Lemma 1** Suppose that \( g \) is nonincreasing. Further, suppose that \( \beta_j \) is nondecreasing and that for any \( v \in (0, 1) \),

\[
\beta'_j(v) = 0 \Rightarrow \beta_j(v) = 0. \tag{6}
\]

Then, \( R_i(v) < \beta_j(v) \) if \( \beta_j(v) > 0 \).

**Lemma 2** Suppose that \( g \) is nonincreasing and that for some measurable \( \hat{V} \equiv (\hat{v}, \bar{\nu}) \subset [0, 1] \), \( \beta_2(v) = \hat{\beta}_2 > 0 \) for all \( v \in \hat{V} \). Then, there exists some \( v^* > \hat{v} \) and \( R_1(v) < \hat{\beta}_2 \) for all \( v \in (\hat{v}, v^*) \).

**Lemma 3** Suppose that \( g \) is nonincreasing. A strategy profile \( (\beta_1, \beta_2) \) does not constitute an equilibrium if \( \beta_1 \) and \( \beta_2 \) are nondecreasing, and if for some measurable \( \hat{V} \subset [0, 1] \),

\[
\forall v \in \hat{V}, \beta_j(v) = \hat{\beta}_j > 0,
\]

and if, for some \( v \in \hat{V} \), \( \beta_i(v) > 0 \).
Lemma 2 is used to prove lemma 3. If lemma 3 is true, there is no measurable $\hat{V}$ where $\beta_i$ is constant in the equilibrium. Therefore, it is sufficient to consider increasing strategies, which satisfy the condition of lemma 1. Lemma 1 shows $\beta_i(v) \geq \beta_j(v) > 0$ cannot be an equilibrium for any $v \in [r, 1)$. Since we consider $\beta_i$ is nondecreasing, $\beta_1(v) = 0$ or $\beta_2(v) = 0$ must hold in an equilibrium. The remainder of the proof is to show the lemmas.

**Proof of Lemma 1.** Let $(i, j) = (1, 2)$ without loss of generality. Let $\pi_1(b_1, v)$ be bidder 1’s associated expected payoff, where bidder 1 with her value $v$ chooses a strategy $\beta_1(v) = b_1$ and where bidders 2 follows a strategy $\beta_2(v_2)$. Then,

$$
\pi_1(b_1, v) = v \left[ \int_{b_1 < \beta_2} v \int_{b_1 + v_2} dG(w_3)dF(v_2) + \int_{b_1 > \beta_2} \int_0^{v + \beta_2} dG(w_3)dF(v_2) \right] \\
- \int_{b_1 > \frac{1}{2}w_3} \int_{\beta_2 > \frac{1}{2}w_3} \frac{1}{2} w_3 dF(v_2) dG(w_3) \\
- b_1 \int_{b_1 < \beta_2} \int_{b_1 + v_2} dG(w_3)dF(v_2) \\
- \int_{b_1 > \beta_2} \int_{\frac{v + \beta_2}{2 \beta_2}} (w_3 - \beta_2) dG(w_3)dF(v_2). 
$$

(7)

Suppose that bidder 2’s strategy $\beta_2$ has $(\beta_2)^{-1}$ and $\beta_2^2 > 0$. Let $\bar{b} \equiv \beta_2(1)$. If bidder 1 chooses $b_1 > \bar{b}$, bidder 2 stops on ahead with probability 1. Hence, bidder 1’s expected payoff is constant for all $b_1 \geq \bar{b}$. Let $\psi \equiv (\beta_2)^{-1}$. By differentiating (7) with respect to $b_1$, we have

$$
\frac{\partial \pi_1}{\partial b_1} = (v - b_1) \int_{b_1 < \beta_2} g(b_1 + v_2)dF(v_2) - \int_{b_1 < \beta_2} \int_{b_1 + v_2} dG(w_3)dF(v_2) \\
- \psi'(b_1) f(\psi(b_1)) \left[ (v - b_1) \{ G(b_1 + \psi(b_1)) - G(v + b_1) \} + \int_{2b_1}^{v + b_1} (w_3 - 2b_1) dG(w_3) \right]. 
$$

(8)
Suppose \( \beta_2(v) \leq b_1 \leq \bar{b} \). Since \( \psi(b_1) \geq v \) and \( g \) is nonincreasing,

\[
(v - b_1) \int_{b_1 < \beta_2} g(b_1 + v) dF(v) - \int_{b_1 < \beta_2} \int_{2b_1}^{b_1 + v_2} dG(w_3) dF(v_2) \\
\leq (v - b_1) \int_{b_1 < \beta_2} g(b_1 + v) dF(v) - \int_{b_1 < \beta_2} (v_2 - b_1) g(b_1 + v_2) dF(v_2) \\
= \int_{\psi(b_1) < v_2} (v - v_2) g(b_1 + v_2) dF(v_2) \\
< 0.
\]

Further,

\[
\psi(b_1) < v_2 \\
\psi(b_1) < v_2 (v - v_2) g(b_1 + v_2) dF(v_2) \\
\leq (v - b_1) \int_{b_1 < \beta_2} g(b_1 + v_2) dF(v_2) - \int_{b_1 < \beta_2} (v_2 - b_1) g(b_1 + v_2) dF(v_2) \\
= \int_{\psi(b_1) < v_2} (v - v_2) g(b_1 + v_2) dF(v_2) \\
< 0.
\]

Since \( \psi'(b_1) > 0 \), by (9) and (10), the marginal payoff of bidder 1

\[
\frac{\partial \pi_1}{\partial b_1}(b_1, v) < 0,
\]

for all \( b_1 \in [\beta_2(v), \bar{b}] \). Therefore \( b_1 \geq \beta_2(v) \) is never a best response.

Next, suppose that \( \beta_2 \) has a jump at \( \hat{v} \in V \). Suppose that \( \beta_2(v) \to \beta \) as \( v \to \hat{v}^- \), and that \( \beta_2(v) \to \bar{\beta} \) as \( v \to \hat{v}^+ \) (\( \beta_2^* < \bar{\beta} \)). The marginal payoff of bidder 1 in the jump interval is expressed as (8) with \( \psi'(b_1) = 0 \). Therefore, \( \frac{\partial \pi_1}{\partial b_1}(b_1, \hat{v}) < 0 \) for \( b_1 > \bar{\beta} \). In addition, since \( \psi'(\beta - \epsilon) > 0 \) for small \( \epsilon > 0 \),

\[
\frac{\partial \pi_1}{\partial b_1}(b_1, \hat{v}) \to \xi < 0
\]
as \( b_1 \to \bar{\beta}^- \). Therefore, \( R_1(\hat{v}) < \bar{\beta} \).

**Proof of Lemma 2.** If \( \beta_2 \) has multiple constant areas, choose the highest one.

We let \( \hat{V} = (\hat{v}, 1] \) without loss of generality. This is because of the following consideration. Suppose \( \hat{v} < 1 \). For any \( v_1 \in \hat{V} \), \( b_1 > \hat{\beta}_2 \) is never a best response since the marginal payoff is negative by Lemma 1. Hence, it is sufficient to consider \( 0 \leq b_1 \leq \hat{\beta}_2^+ \) when we consider bidder 1’s best response. As long as \( b_1 \in [0, \hat{\beta}_2^+] \), the expected payoff conditional on the events such that \( v_2 \in (\hat{v}, 1] \) is indifferent from \( b_1 \). Therefore, we can subtract it and ignore the events when \( v_2 \in (\hat{v}, 1] \).

Let \( \hat{V} = (\hat{v}, 1] \) and \( v \in \hat{V} \). First, consider the case where \( b_1 = \hat{\beta}_2^+ \). Then the bidder
1’s expected payoff is
\[
\pi_1(\hat{\beta}_2^+, v) = \int_0^1 \int_0^{v+\beta_2} dG(w_3) dF(v_2) - \int_0^{2\beta_2} \int_{2\beta_2 > w_3} \frac{1}{2} w_3 dF(v_2) dG(w_3)
\]
\[ - \int_0^1 \int_{2\beta_2}^{v+\beta_2} (w_3 - \beta_2) dG(w_3) dF(v_2). \tag{12}
\]
On the other hand, when \( b_1 = \hat{\beta}_2^- \), bidder 1’s interim expected payoff is
\[
\pi_1(\hat{\beta}_2^-, v) = \int_0^{\hat{\beta}_2^-} \int_0^{v+\beta_2} dG(w_3) dF(v_2) + \int_{\hat{\beta}_2^-}^1 \int_0^{\hat{\beta}_2^+, v} dG(w_3) dF(v_2)
\]
\[ - \int_0^{2\beta_2} \int_{2\beta_2 > w_3} \frac{1}{2} w_3 dF(v_2) dG(w_3) - \hat{\beta}_2 \int_{\hat{\beta}_2^-}^{\hat{\beta}_2^+, v} dG(w_3) dF(v_2)
\]
\[ - \int_{\hat{\beta}_2^-}^{\hat{\beta}_2^+, v} (w_3 - \beta_2) dG(w_3) dF(v_2). \tag{13}
\]
With some calculations, we have
\[
\pi_1(\hat{\beta}_2^+, v) - \pi_1(\hat{\beta}_2^-, v) \leq (v - \hat{\beta}_2) \int_{\hat{\beta}_2}^1 (G(v + \hat{\beta}_2) - G(v_2 + \hat{\beta}_2)) dF(v_2). \tag{14}
\]
If we take some \( v^* > \hat{v} \) such that is sufficiently close to \( \hat{v} \), the right hand side of (14) is negative. Hence, we have for all \( v \in (\hat{v}, v^*) \),
\[
\pi_1(\hat{\beta}_2^+, v) < \pi_1(\hat{\beta}_2^-, v) < \pi_1(\hat{\beta}_2^-, v).
\]
\[ \therefore R_1(v) < \hat{\beta}_2. \]

For other constant areas, \( R_1 \leq \hat{\beta}_2 \) also holds by the similar consideration. \( \square \)

**Proof of Lemma 3.** Let \( \hat{V} = (\hat{v}, \hat{v}) \). Suppose that for all \( v \in \hat{V} \), \( \beta_2(v) = \hat{\beta}_2 > 0 \). Then, there exists some measurable \( V^* \subset \hat{V} \), and for all \( v \in V^* \), \( R_1(v) < \hat{\beta}_2 \) by Lemma 2. If \( \beta_1 \) and \( \beta_2 \) constitute an equilibrium, they are the best responses to each other.

**Case 1:** \( R_1'(v) > 0 \) for some \( v \in V^* \).

By the same argument in Lemma 1, \( R_2(v) < R_1(v) \). Therefore, \( R_2(v) \neq \hat{\beta}_2 \).

**Case 2:** \( R_1'(v) = 0 \) for all \( v \in V^* \) and \( R_1 \) jumps over \( \hat{\beta}_2 \) at \( v^* = \sup_{v \in V^*} v \).

A consideration similar to Lemma 1 implies \( R_2(v^*) < \hat{\beta}_2 \).

**Case 3:** \( R_1'(v) = 0 \) for all \( v \in (\hat{v}, 1] \).

Suppose that \( 0 < R_1(v) = \hat{\beta}_1 < \hat{\beta}_2 \) for all \( v > \alpha \), where \( \alpha \leq \hat{v} \). Then, by Lemma 2, there exists some \( \gamma > \alpha \) and for all \( v \in (\alpha, \gamma] \), \( R_2(v) < \hat{\beta}_1 \). However, \( R_1(\gamma) = \hat{\beta}_1 \) does not hold by Lemma 1. \( \blacksquare \)
C.2 Proof of Theorem 3

It is obvious that global bidders have the dominant strategy of truthful reporting. For each local bidder, it is also obvious that he bids truthfully when he is the unique local bidder. When more than \( k \) local bidders are left, each local bidder has to continue bidding in order to obtain the item. Bidding over true values brings bidders in negative payoffs. Therefore, truthful bidding is an optimal strategy for each local bidder as long as more than \( k \) local bidders continue to bid.

The remainder of the proof is to show the third term of the theorem. Let \( w^{(h)} \) be the \( h \)-th order statistics of the global bidders’ values from highest to smallest. Suppose that at a price vector \((s, \ldots, s)\), the \((k + 1)\)-th highest local bidder stops bidding and only \( k \) local bidders are left. Let bidders 1, \ldots, \( k \) be the remaining bidders without loss of generality. Further, suppose that there are \( l \) (\( \leq n \)) global bidders standing at \((s, \ldots, s)\).

Then, consider the strategy for local bidders after \((s, \ldots, s)\).

Suppose that all the bidders except bidder \( i \) stop immediately at \((s, \ldots, s)\) regardless of values as long as more than one local bidders remain. If bidder \( i \) stops immediately at \( s \) and if it is accepted, then \( i \)'s interim expected payoff at \((s, \ldots, s)\) is

\[
(v_i - s) \Pr\{v_j + (k-1)s > w^{(1)}|w^{(l)} > ks, v_j > s\} = \frac{v_i - s}{1 - F(s)} \int_s^1 H_l(v_j + (k-1)s|w^{(l)} > ks) dF(v_j),
\]

where \( H_l(w^{(1)}|Z) \) denotes the conditional distribution of the first order statistic \( w^{(1)} \) among \( l \) valuations. On the other hand, if bidder \( i \) does not stop at \( s \), she consequently bids up truthfully since all the other bidders stops. Hence, the interim expected payoff is given by

\[
\int_{ks}^{v_i+(k-1)s} (v_i + (k-1)s - w^{(1)}) dH_l(w^{(1)}|w^{(l)} > ks) = \int_{ks}^{v_i+(k-1)s} H_l(w^{(1)}|w^{(l)} > ks) dw^{(1)}.
\]

The equality follows from integration by parts. Therefore, stopping at \( s \) is optimal if

\[
\frac{v_i - s}{1 - F(s)} \int_s^1 H_l(v_j + (k-1)s|w^{(l)} > ks) dF(v_j) \geq \int_{ks}^{v_i+(k-1)s} H_l(w^{(1)}|w^{(l)} > ks) dw^{(1)}
\]

for each \( l = 1, \ldots, n \), all \( s < 1 \), and all \( v_i > s \). The left and right hand sides of (17) are linear and convex functions of \( v_i \) respectively. Since (17) holds when \( v_i = s \), it is
equivalent to
\[
\frac{1 - s}{1 - F(s)} \int_{s}^{1} H_{l}(v_{j} + (k - 1)s | w^{(l)} > ks) dF(v_{j}) \geq \int_{ks}^{1 + (k - 1)s} H_{l}(w^{(1)} | w^{(l)} > ks) dw^{(1)}
\]
for each \( l = 1, \ldots, n \) and all \( s \in [r, 1) \). Note that
\[
H_{l}(x | w^{(l)} > ks) = \frac{1}{(1 - G(k s))} \left\{ G(x) - G(k s) \right\}^{l}.
\]
Therefore, (18) yields
\[
\int_{s}^{1} \left\{ G(x + (k - 1)s) - G(k s) \right\}^{l} \left\{ \frac{1 - s}{1 - F(s)} f(x) - 1 \right\} dx \geq 0.
\]
The inequation holds if \( F \) is the uniform distribution. ■

References


