The Weakest Link, Condorcet Consistency, and Sequential vs. Simultaneous Voting

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(Incomplete draft)

Abstract

The weakest link voting selects a single winner from a number of candidates by successive eliminations. The voting takes place in rounds and in each round the candidate receiving the smallest number of votes – the weakest link – is eliminated. If the voters vote strategically, the weakest link rule will select the Condorcet winner – when it exists – to become the ultimate winner. The voting rule is shown to be part of a larger family of sequential voting rules, along with the more familiar binary voting, that share this special feature. In contrast, some of the popular one-shot voting rules such as plurality rule, approval voting, Borda rule and negative voting fail to be Condorcet consistent. The equilibrium properties of the weakest link voting, in the absence of a Condorcet winner, are also analyzed. JEL Classification Numbers: P16, D71, C72.

Key Words: Sequential voting, Condorcet winner, weakest link voting, binary voting, Banks set, Markov equilibrium, complexity aversion.

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1 Introduction

An important criterion in assessing the desirability of any voting rule is whether it induces the Condorcet winner, when the latter exists.\footnote{A candidate is a Condorcet winner if it wins over every other candidate by majority vote.} In fact, any failure of its attainability is generally considered to be a symptom of the various instabilities that often plague many voting mechanisms. Given its importance, the attainability of the Condorcet winner has come to be known as the Condorcet consistency property that any voting rule might be expected to satisfy.\footnote{However, as it is well-understood, whether a voting rule attains a desirable outcome depends as much on the voting rule as on the chosen equilibrium solution concept.} In this paper, we will explore the issue of Condorcet consistency, besides some other issues, in the context of a so far neglected voting mechanism – the weakest link voting – and a number of other popular voting rules. Given its relative obscurity, it is imperative that we explain the weakest link voting first.

In the contest for the leadership of the Conservative Party in Great Britain, a small number of candidates who put themselves up for selection are all simultaneously voted upon by the party’s parliamentary members in a sequence of rounds. In each round, the candidate getting the smallest number of votes is eliminated. The surviving candidates then proceed to the next round where a similar vote is taken. This process continues until only two candidates remain who then face the poll of a large number of party members. We call this elimination procedure\footnote{See the explanation by Julian Glover in The Guardian, July 10, 2001 accessible at http://politics.guardian.co.uk/Print/0,3858,4196604,00.html. This voting procedure is only a recent innovation in democratically electing the party leader; on all other previous occasions, the Conservative Party leader used to be chosen on an ad hoc basis.} the weakest link rule.\footnote{Incidentally, there is also a popular TV quiz show on BBC, called The Weakest Link, which roughly follows a similar procedure of elimination. Also, followers of various game shows on television would surely recognize implicit or explicit use of the weakest link feature.}

We will consider strategic voting in the successive elimination of candidates of the weakest link procedure.\footnote{The idea of strategic voting, as opposed to sincere voting, was popularized by Farquharson (1969) and is also known as sophisticated voting.} Simultaneous consideration of all the remaining candidates in each voting round with an eye also to how the voters might vote in the future rounds, makes the analysis of strategic voting in the weakest link game extremely complex. However, we are able to make sharp predictions when there
is a Condorcet winner. We show that the unique equilibrium of the weakest link game will select the Condorcet winner. In the absence of a Condorcet winner, our analysis yields only partial information regarding the likely nature of equilibrium outcomes.

In analyzing weakest link voting we propose a new equilibrium solution procedure combining subgame perfection and a version of non-domination along subgames, for backward eliminations of strategies. Also, we impose the behavioral assumption that the voters use only Markov strategies that can be justified if the voters are complexity averse. The procedure differs from the usual procedure of iterative elimination of weakly dominated strategies used elsewhere in the voting literature (Moulin, 1979) in some key aspects that we discuss later in the paper. While iterative elimination of Moulin (1979) and others fail to cope with the multiplicity of voting equilibria in the weakest link game, our refinement proves powerful in restricting the equilibrium set at least when there is a Condorcet winner. We hope that our solution procedure will also be useful in other contexts beyond the specific voting game analyzed in this paper.

Condorcet consistency of the weakest link game compares nicely with the same feature of another widely-studied voting rule – the sequential binary voting using majority rule. In the case of binary voting the candidates contest only in pairs, as opposed to all (surviving) candidates contesting simultaneously in each round in the weakest link voting. Banks (1985) completely characterized the equilibria of the binary voting for arbitrary ordering of candidates and the equilibrium set has become known as the Banks set. Our equilibrium characterization of the weakest link voting is compared with Banks’ characterization. We show that the weakest link rule enlarges the set of equilibrium outcomes relative to the Banks set.

We also address the question whether Condorcet consistency would hold for other important voting rules such as plurality rule, approval voting, Borda rule and negative voting, when the voters vote strategically. We show that none of these rules are Condorcet consistent. Given that these voting mechanisms are among the better-known procedures in making important decisions in a multitude of contexts, failure of Condorcet consistency comes as a disturbing feature that we feel should be noted. In fact, the two sets of observations – Condorcet consistency of the weakest link and the binary voting, and its failure in the case of the above mentioned one-shot voting rules – indicates that sequential voting mechanisms tend to per-
form better than one-shot simultaneous voting mechanisms. So we consider a class of sequential voting mechanisms embracing both the weakest link and the binary voting. In these mechanisms, candidates are eliminated successively based on the voters’ freshly declared ranking of the (remaining) candidates at each stage. We identify a rather mild sufficient condition for Condorcet consistency: a candidate receiving a majority of the top rank may not be eliminated. This condition will help to clarify the contrast between the specific sequential voting rules satisfying Condorcet consistency and their close analogues in one-shot voting that fail Condorcet consistency.

The paper is organized as follows. In the next section we formally describe the voting rules and the related equilibrium solution concepts. Section 3 forms the core of the paper with the results on sequential voting. In section 4, some popular one-shot voting mechanisms are analyzed. Section 5 concludes. Some of the proofs are included in an Appendix.

2 The Voting Rules and Equilibrium Solutions

The weakest link

Formally, the weakest link voting proceeds as follows. The voting takes place in a finite number of stages. The set of candidates is denoted as $K$ with cardinality $k$, and the voter set is denoted as $N$ with cardinality $n$, both $k$ and $n$ at least three, and $K \cap N = \emptyset$. Each voter has a strict preference ordering over the candidates. A single winner is selected after completion of $k - 1$ voting stages. In stage 1, the starting point of the weakest link game, all candidates are simultaneously voted upon by the voters. The candidate receiving the smallest number of votes drops out from the competition and the remaining $k - 1$ candidates proceed to a similar stage 2 voting; any tie is broken by a single deterministic tie-breaking rule ranking all $k$ candidates. This procedure continues until all but one of the candidates have

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6Sequential vs. simultaneous voting in this paper is different from other uses of the terminologies in the literature, e.g., Dekel and Piccione (2000) and Strumpf (2002). In ours, the key point is whether the candidates are eliminated sequentially or simultaneously and its impact on the voting outcome; in Dekel and Piccione, voters with different information can either cast their votes simultaneously or sequentially and the main issue is information aggregation; Strumpf’s paper is about how outcomes of early elections influence later elections with different electorates.
been eliminated.\footnote{Taking the elimination all the way is only one step beyond the actual elimination procedure of the aforementioned Conservative Party Leadership contest. There, in the final round, the voters set was significantly altered – from 166 MPs to around 300,000 – by including many general party members, thus making the voting environment “large” which is a standard way of aggregating information. We do not address the information aggregation issue.}

**The equilibrium**

The equilibrium outcome of the weakest link voting follows successive eliminations of strategies failing subgame perfection and a special type of non-domination. In addition, voters will be assumed to use only Markov strategies. These are explained next.

In any final stage subgame (i.e., at stage $k - 1$), the voters’ strategies must be a Nash equilibrium and must not be (weakly) dominated when considered specifically with respect to the subgame;\footnote{While a particular strategy may not be dominated in the entire game, it could still be dominated when restricted to a (smaller) subgame.} all other strategies are eliminated. Then in the subgames starting with stage $k - 2$, only the strategies that survived eliminations at stage $k - 1$ are considered. In any of these subgames, again, the voters’ strategies from the restricted set must be a Nash equilibrium and must not be dominated along the subgame, where the permissible strategies of the voters with respect to which the weak-domination check is carried out are the strategies that have survived backward eliminations up to that stage. We follow this backward-elimination procedure all the way to stage 1. In general, in the subgames following a stage, irrespective of whether such subgames are reached or not, no voter will use strategies that fail to survive backward eliminations.

The voters adopt only Markov strategies, that is, the strategies at any stage onwards depend only on the candidates who have survived up to that stage and not on the specific history leading up to it.

We now develop the equilibrium procedure more formally.

The voters’ strategies are easier to describe with respect to histories. A history, $h$, associated with any particular node at any stage of the successive elimination process is a complete description of the actual voting decisions leading up to that node. Given a deterministic tie-breaking rule, any history $h$ uniquely defines a
subgame $\Gamma(h)$ determining the set of candidates $C \subseteq K$ who have survived at the end of $h$; $\Gamma(h_0)$ denotes the entire game where $h_0$ denotes the null history.

Let $\mathcal{H}_C = \{ h | C \text{ is the set of remaining candidates} \}$, and $\mathcal{H} = \bigcup_{C \subseteq K} \mathcal{H}_C$ be the set of all histories.

Define the set of (pure) strategies of voter $i$ by $S_i = \{ s_i : \mathcal{H} \to K \text{ s.t. } s_i(h) \in C \text{ if } h \in \mathcal{H}_C \}$.

For any $h$, let $C(h)$ be the set of candidates left with cardinality $n(h)$, and $S_i(h)$ be the restriction of $S_i$ to the subgame $\Gamma(h)$. We want to define $S(h) = \Pi_i S_i(h)$ to be the product set of strategies in the subgame $\Gamma(h)$. Thus,

$$
\text{if } s_i \in S_i(h),
\text{then } s_i : \tilde{\mathcal{H}}(h) \to C(h) \text{ s.t. } s_i(h') \in C' \quad \forall h' \in \mathcal{H}_C \cap \tilde{\mathcal{H}}(h) \neq \emptyset
$$

where $\tilde{\mathcal{H}}(h)$ is the set of histories that follow $h$.

Following any history $h$, the equilibrium solution $s(h)$ for the subgame $\Gamma(h)$ will be defined inductively as follows.

At any history $\tilde{h}$ s.t. $n(\tilde{h}) = 2$, the solution $s$, abbreviated for $s(\tilde{h})$, is a Nash equilibrium (N.E.) in the subgame $\Gamma(\tilde{h})$ and not weakly dominated in this subgame. That is,

$$
\text{N.E. \qquad } \pi_i(s_i, s_{-i}) \geq \pi_i(s'_i, s_{-i}) \quad \forall s'_i \in S_i(\tilde{h});
$$

(Weak non-domination) \qquad $\exists s'_i \in S_i(\tilde{h}) \text{ s.t. }$

$$
\pi_i(s'_i, s'_{-i}) \geq \pi_i(s_i, s'_i) \quad \forall s'_{-i} \in s_{-i}(\tilde{h}), \text{ and }
\pi_i(s'_i, s'_{-i}) > \pi_i(s_i, s'_i) \text{ for some } s'_{-i} \in S_{-i}(\tilde{h}).
$$

Suppose $s(h')$ is defined for all $h'$ s.t. $n(h') \leq j - 1$. Then for any $h$ with $j$ candidates left, the solution $s(h)$ will satisfy the following:

(i) \qquad $s(h) \in S(h)$ is a N.E. of $\Gamma(h)$;

(ii) \qquad $\exists s'_i \in S_i(h) \text{ s.t. }$

$$
\pi_i(s'_i, s'_{-i}) \geq \pi_i(s_i, s'_{-i}) \quad \forall s'_{-i} \in S_{-i}(h) \text{ s.t. } s'_{-i}(h') = s_{-i}(h') \quad \forall h' \in \hat{\mathcal{H}}(h), \text{ and }
\pi_i(s'_i, s'_{-i}) > \pi_i(s_i, s'_{-i}) \text{ for some } s'_{-i}.
$$

By iterating backwards all the way to the first stage, we obtain our equilibrium solution, $s(h_0)$, to be referred simply as equilibrium.
**Definition 1** We define an equilibrium, \( s(h_0) \), to be a **Markov equilibrium**, if the voters are restricted to use only Markov strategies. That is, each \( i \) selects strategies only from

\[
S^M_i = \{ s_i : \mathcal{H} \to \mathcal{K} \text{ s.t. } \forall h \in \mathcal{H}_C, s_i(h) \in C; \text{ and } \]

if \( C(h) = C(h') \) then \( s_i(h) = s_i(h') \}.
\]

This completes the formal equilibrium procedure.

**Remarks.** Our backward-elimination procedure differs, it must be noted, from the more familiar procedure of iterative elimination of (weakly) dominated strategies in the existing voting and game-theory literature, in one important aspect: while in the latter approach the weak-domination check is carried out in relation to the whole game, ours is only along the subgames.\(^9\) Iterative elimination on its own, or even in combination with subgame perfection, is unlikely to solve the miscoordination problems that result in undesirable outcomes.\(^10\) It is well-known in other voting contexts that iterative elimination can have very little elimination power – for example, Dhillon and Lockwood (forthcoming) have shown in the case of plurality voting (see their Lemma 1 and the related discussion) that strategies of voting *any* candidate other than one’s lowest-ranked candidate will survive iterative eliminations of weakly dominated strategies.

As such there is no guarantee that \( s(h_0) \) will be single-valued, so that multiplicity remains a distinct possibility. However, in general, uniqueness of solution is very difficult to guarantee even for other major equilibrium refinement approaches.

To overcome the difficulties due to the multiplicity of equilibrium solution, our equilibrium procedure should be considered in conjunction with the behavioral assumption of Markov strategies. The theoretical justification for Markov strategies in general dynamic games is mostly lacking. However, in the specific context of our weakest link voting, Markov strategies can be justified (Theorem 2). Clearly, imposing the Markov property can only serve to (weakly) reduce the equilibrium set.

\(^9\)Moulin (1979) formally analyzed the iterative elimination procedure to generalize the concept of sophisticated voting and applied it to a significant class of voting – voting by veto, kingmaker and voting by binary choices. A recent paper by Dhillon and Lockwood (forthcoming) also employs the procedure in the case of plurality voting.

\(^10\)We thank Bhaskar Dutta for bringing this point to our attention.
Weakest link and binary voting: Comparison

An appropriate comparison would be between the weakest link rule and the sequential binary voting, the latter having received a great deal of attention in the context of agenda formation. In binary voting, candidates are first ordered in a sequence and then pairwise comparisons are made using the majority rule. The candidate to survive the sequential elimination process in binary comparisons is declared the winner.\(^{11}\) The key differences between the sequential binary voting and the weakest link voting are as follows:

- while in the former at any single stage a comparison is made between a pair of candidates, in the latter all (remaining) candidates are simultaneously compared;
- the order in which the candidates are voted in the binary voting is often crucial in determining the ultimate winner and thus the voting rule treats the candidates asymmetrically, whereas in the weakest link voting all candidates are treated equally (except in the event of a tie when a deterministic tie-breaker determines which candidate is to drop out).

Thus, besides the difference in contexts where the respective two voting rules might apply,\(^{12}\) on the criterion of fairness the weakest link voting has an advantage over the binary voting.

One-shot voting rules

To contrast the weakest link rule, we will also consider various other voting rules such as plurality rule, approval voting, Borda voting and negative voting. For all of these one-shot simultaneous voting rules the equilibrium solution concept is undominated Nash, i.e., the voters’ strategies must be a Nash equilibrium and must not be (weakly) dominated. Note that our twin requirements of subgame perfection and non-domination (independently of the Markov strategy assumption) boil down to the equilibrium definition for the one-shot voting rules. Thus, the comparisons to be made in section 3 between the weakest link voting and the one-shot voting rules are based on the same benchmark solution concept. However, the sequential

\(^{11}\)A complete characterization of the equilibrium of this sequential binary voting appears in Banks (1985), following up on partial characterizations by Miller (1980) and Shepsle and Weingast (1984).

\(^{12}\)It must be noted that the two voting rules are suitable only for small elections, i.e., elections involving a small number of voters.
structure of the weakest link voting would give us additional leverage that will be missing from the one-shot voting rules.

3 Sequential voting

3.1 Condorcet consistency

We start with some key results on the weakest link rule and generalize to a class of sequential voting.

**Lemma 1** Suppose the voters are sincere. Then the weakest link voting rule is not Condorcet consistent.

**Proof.** Consider the following strict preference ordering of 4 candidates by 3 voters:

1: \( y, x, z, w \)
2: \( z, x, y, w \)
3: \( w, x, z, y \).

The weakest link voting will select either \( y \) or \( z \) or \( w \) depending on the tie-breaking rule, whereas \( x \) is the Condorcet winner. \( \Box \)

**Theorem 1** Suppose the voters are strategic. Assuming that the voters use only Markov strategies, the weakest link voting with a deterministic tie-breaking rule is Condorcet consistent.

**Proof.** Suppose there is a Condorcet winner \( z \), but the weakest link voting game has a Markov equilibrium that results in some other candidate \( z_1 (\neq z) \) as the ultimate winner. Order the stages of elimination of \((k - 1)\) candidates as follows:

<table>
<thead>
<tr>
<th>The winner</th>
<th>( z_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stage k-1</td>
<td>( z_1, z_2 )</td>
</tr>
<tr>
<td>Stage k-2</td>
<td>( z_1, z_2, z_3 )</td>
</tr>
<tr>
<td>Stage k-3</td>
<td>( z_1, z_2, z_3, z_4 )</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>Stage 1</td>
<td>( z_1, z_2, z_3, z_4, \ldots, z_k )</td>
</tr>
</tbody>
</table>
Thus in stage $j$, the eliminated candidate is labelled as candidate $z_{k-j+1}$, $j = 1, 2, \ldots, k - 1$.

Initially, consider stage $k - 1$ and suppose $z_2 = z$. Given that $z_2$ is the Condorcet winner, clearly for any voter who prefers $z_2$ over $z_1$, and there will be a majority of such voters, the strategy of choosing $z_1$ is weakly dominated in the stage-$(k - 1)$ subgame: choosing $z_2$ instead will produce no worse and sometimes a better outcome (when $z_2$ wins the majority vote). (In fact, sincere voting by all voters is the only Nash equilibrium that is also undominated in this final stage subgame.) Thus if the Condorcet winner $z$ survives up to stage $k - 1$, she must be the ultimate winner so that $z_1 = z$; $z_2 = z$ is not possible.

Now suppose the following hypothesis is true:

*If candidate $z$ survives up to stage $j$ on- or off-the-equilibrium path, then she will also survive the remaining stages and become the ultimate winner.*

We then prove that having proceeded to any stage-$(j - 1)$ subgame with only $k - j + 2$ candidates left, $z$ will also survive stage $j - 1$ to move up to stage $j$ and thus become the ultimate winner.

Suppose on the contrary that $z_{k-j+2} = z$. Consider those voters who prefer $z_{k-j+2}$ over $z'_1$, where $z'_1$ is going to be the ultimate winner if $z$ is eliminated in stage-$(j - 1)$ voting. By definition of $z$, these voters will form a majority. Consider any such voter’s strategy in the stage-$(j - 1)$ subgame, where $z'_1$ is to become the ultimate winner. Suppose the representative voter were to choose some $z' \neq z$. This, we claim, is not possible. If the voter switches his vote from $z'$ to $z$ and $z$ is not eliminated, which will be the case if all who prefer $z$ over $z'_1$ vote for $z$, then $z$ survives, by hypothesis, all the subsequent stages and becomes the ultimate winner that is better than $z'_1$; if, on the other hand, $z$ is eliminated then by the Markov property of the voters’ strategies $z'_1$ becomes the ultimate winner. Thus, in the subgame $z$ will weakly dominate $z'$ so that the representative voter must vote for $z$ only. This implies a majority of voters would vote for $z$, contradicting that $z_{k-j+2} = z$.

We already proved our hypothesis for $j = k - 1$. So use an induction argument to conclude that the Condorcet winner $z$ will be the ultimate winner in the weakest

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13 This subgame can be on- or off-the-equilibrium path.
14 $z'_1 = z_1$ if this subgame is on the equilibrium path, and otherwise $z'_1$ can be some other candidate.
link voting game. Our argument so far does not make any reference to the tie-breaking rule, thus the weakest link voting is Condorcet consistent for any arbitrary deterministic tie-breaking rule.  

**Q.E.D.**

**Remarks.** The procedure of eliminating the weakest link, if extended naturally to eliminate all but one candidate in a single round of voting (as opposed to successive eliminations), translates into the one-shot plurality voting rule. The plurality voting will be later shown to fail Condorcet consistency (Proposition 4). Thus, Theorem 1 illustrates the distinct advantage of the sequential elimination procedure over the one-shot elimination procedure (of the plurality rule). Later (see the discussion following Theorem 3), a similar comparison will be made between a sequential veto procedure and a one-shot veto procedure.

A weaker version of the Markov property would suffice for Theorem 1 proof. All we require is that the strategies do not depend on the history through the specific configuration of votes that led to the particular candidates’ eliminations. However, the strategies can still depend on the order in which the candidates were eliminated. In fact, if we assume that the votes are not revealed between stages but only the identity of the eliminated candidate at each stage is announced, then we do not need the Markov property.  

Admittedly, the assumption that the voters use only Markov strategies could be considered a limitation of our Condorcet consistency result for the weakest link voting. So we now provide a formal justification behind the behavioral assumption.

Recall $S_i$ as the strategy set of voter $i$ with $s_i: \mathcal{H} \rightarrow \mathcal{K}$ s.t. $s_i(h) \in C \ \forall h \in \mathcal{H}_C$. Also, let $S = \Pi_i S_i$.

**Definition 2** A strategy $s_i \in S_i$ is **more complex** than another strategy $s'_i \in S_i$ if \exists C s.t.

(i) $s_i(h) = s'_i(h) \ \forall h \notin \mathcal{H}_C$;

(ii) $s'_i(h) = s'_i(h') \ \forall h, h' \in \mathcal{H}_C$;

(iii) $s_i(h) \neq s_i(h')$ for some $h, h' \in \mathcal{H}_C$.

The above ordering of complexity is only a partial ordering. Nevertheless, it will prove a powerful one for our purpose. Based on this ordering let us introduce an equilibrium definition, which is a further refinement of our earlier definition of **equilibrium solution**.
**Definition 3** A strategy profile $s \in S$ will be called a **simple equilibrium** of the weakest link voting game, if

(i) $s$ is an equilibrium solution of $\Gamma(h_0)$;

(ii) $\not\exists i \in N$ s.t. $\exists s'_i \in S_i$ s.t. $\pi_i(s'_i, s_{-i}) = \pi_i(s_i, s_{-i})$, and $s_i$ is more complex than $s'_i$, where $\pi_i(\cdot, \cdot)$ is the payoff function of voter $i$.

Note that while the definition of simple equilibrium allows history-dependent (i.e., non-Markov) strategies, the second condition reflects the implicit assumption that the voters are averse to complexity (as in Definition 2) unless it helps to increase their payoffs. Thus, simplicity of the simple equilibrium is a very weak, and in our view plausible, requirement for any descriptive analysis. We will therefore use the simplicity criterion for equilibrium selection.

**Theorem 2** Any simple equilibrium is also a Markov equilibrium.

*Proof.* Suppose $s \in S$ is a simple equilibrium but not a Markov equilibrium. Then there exists some $i, C$ and $h, h' \in H_C$ s.t. $s_i(h) \neq s_i(h')$. Clearly, if $H_C \cap E \neq \emptyset$ where $E$ is the equilibrium path corresponding to the simple equilibrium $s$, then $H_C \cap E$ is unique; that is, $C$ happens on the equilibrium path at most once. Now consider another strategy $s'_i \in S_i$ s.t.

\[
s'_i(h) = s_i(h) \quad \forall h \notin H_C;
\]

\[
\forall h \in H_C, \quad s'_i(h) = \begin{cases} 
 s_i(H_C \cap E) & \text{if } H_C \cap E \neq \emptyset, \\
 a \in C & \text{if } H_C \cap E = \emptyset,
\end{cases}
\]

where $a$ denotes any arbitrary element.

It is easy to see that $s'_i$ is simpler than $s_i$. Moreover, because $s'_i$ differs from $s_i$ only for histories in $H_C$ that are off-the-equilibrium path, $(s'_i, s_{-i})$ will result in the same winner as the equilibrium $s$, so that $\pi_i(s'_i, s_{-i}) = \pi_i(s_i, s_{-i})$. Hence, $s$ cannot be a simple equilibrium – a contradiction. Q.E.D.

Thus, so long as one accepts that the voters are complexity averse, by Theorem 2 we are thus able to justify our restriction to Markov strategies in Theorem 1.

Given that the sequential binary voting rule is also Condorcet consistent under strategic voting, Theorem 1 gives reasons to believe that a general sequential

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15 $S^*(h_0)$ does not employ the Markov assumption.

16 See Dutta and Sen (1993), for example.
process of elimination could be the answer to the Condorcet consistency question – an issue to which we turn next.

**Sequential voting.**\(^{17}\) Players vote in rounds. At each stage they simultaneously submit a ranking of the remaining candidates and one candidate is removed. So in stage 1 they submit a ranking of all the candidates and in stage \(k - 1\) they just vote for one of the two surviving candidates. We now specify the following property to define a class of sequential voting rules.

*Non-elimination property:* At any stage when the voters submit their ranking of the remaining candidates, any candidate securing a *majority* of the ‘top’ rank cannot be eliminated.\(^{18}\)

All sequential voting rules satisfying the non-elimination property constitute the family \(\mathcal{F}\). ||

We need to make two clarifications:

First, the general ranking of candidates by voters in the sequential elimination process does not preclude from its domain specific voting rules such as weakest link and binary voting. For the weakest link voting all ranks other than the top rank will be considered of equal (and less) significance, while for the binary voting at any stage only the relevant two candidates’ ordinal ranking will matter,\(^{19}\) and these specifics will be common knowledge when the voters vote.

Second, the non-elimination property defining \(\mathcal{F}\) is quite general. The property merely states which candidate cannot be eliminated in a particular stage, thus leaving open large degrees of freedom for elimination. We deliberately avoid specifying the precise elimination rule (except in the final decision stage) in order to embrace a reasonably wide class of sequential voting rules. The weakest link and the binary voting both satisfy the non-elimination property.

Now go back to the proof of Theorem 1. It is not difficult to see that the arguments there will apply equally for the entire family \(\mathcal{F}\). In particular, the non-elimination property comes to equal effect in sustaining the Condorcet winner as follows. Assume the hypothesis for stage \(j\) is true and consider stage-(\(j - 1\))

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\(^{17}\)By sequential voting, we mean sequential elimination of candidates.

\(^{18}\)This non-elimination property implies *majority rule* in stage \(k - 1\).

\(^{19}\)Thus, in binary voting the two candidates’ originally submitted ranks will be relabelled with the higher ranks assigned as rank 1 and the lower ones as rank 2, where rank 1 is to be considered higher than rank 2.
subgame.\textsuperscript{20} Suppose if \( z \) is eliminated in stage \( j - 1 \) then \( z'_1 \) becomes the ultimate winner. There will be a majority of voters who prefer \( z \) over \( z'_1 \). Suppose any such voter places some candidate \( z' \neq z \) at the top of his ranking. This, we claim, is not possible. If the voter instead places \( z \) at the top and \( z \) is \textit{not} eliminated,\textsuperscript{21} then by hypothesis \( z \) will progress to become the ultimate winner, which is a better outcome than \( z'_1 \); on the other hand, if \( z \) is eliminated then by Markov property \( z'_1 \) is the ultimate winner. Thus, placing some \( z' \neq z \) at the top will be weakly dominated, implying that \( z \) must be placed at the top of a majority voters’ rankings. Hence, \( z \) cannot be eliminated in the stage-(\( j - 1 \)) subgame and an induction argument would induce \( z \) to become the ultimate winner. Thus, we can state the following result:

\textbf{Theorem 3} \textit{All sequential voting rules in the family} \( \mathcal{F} \textit{ will be Condorcet consistent under strategic voting.} \)

Note that while Theorem 3, like Theorem 1, holds only under the Markov assumption, Theorem 2 will continue to have the same force for the sequential family \( \mathcal{F} \). Therefore as in Theorem 1, the Markov assumption is also justifiable for Theorem 3.

To illustrate further the scope of the family \( \mathcal{F} \), consider the following mechanism. \textbf{The sequential veto rule.} According to this rule, at any stage the candidate securing the \textit{largest} number of the bottom rank will be eliminated (with a tie-breaking rule applying appropriately). The bottom rank is assigned the value, 0, and all other ranks (including the top rank) are assigned uniformly the value, 1. The interpretation is that, at each stage a voter will veto one alternative by placing the candidate, whom he least likes to proceed to the next stage, at the bottom of his ranking.

Clearly, the sequential veto rule is within the family \( \mathcal{F} \) and, by Theorem 3, is Condorcet consistent.

The one-shot equivalent of the sequential veto procedure is one where all but one candidate will be eliminated in a single round of voting and the winner is the candidate who collects the \textit{smallest} number of veto – a procedure generally known

\footnote{Clearly, the hypothesis is true for \( j = k - 1 \) because non-elimination property implies majority rule in stage \( k - 1 \).

\footnote{Which will be the case if all who prefer \( z \) over \( z'_1 \) place \( z \) at the top.}
as negative voting. In contrast to the sequential veto rule, the one-shot veto (i.e., negative voting) rule is later shown to fail Condorcet consistency – see Proposition 7. Thus, the sequential elimination renders the veto procedure a clear edge over one-shot elimination.

### 3.2 Further properties of the weakest link rule

Propositions 1–3 will provide some characterizations of the weakest link rule, the limited nature of these results notwithstanding. First, we need some terminologies – for more elaborate descriptions see Dutta, Jackson and Le Breton (2002).

Given the voters’ strict preference orderings over candidates, a binary comparison operator \( T \) defines a candidate \( x \) to be majority preferred over another candidate \( y \), written as \( xTy \), if the number of voters preferring \( x \) over \( y \) exceeds the the number of voters preferring \( y \) over \( x \). The operator \( T \) will be a majority tournament if either \( n \) is odd or a deterministic tie-breaking rule breaks the ties.

A set \( H \subseteq \mathcal{K} \) will be called a chain of \( T \) if \( T \) is a transitive relation with respect to \( H \): the candidates in \( H \) can be ordered such that any earlier candidate in \( H \) beats all other candidates in \( H \) following her.\(^{22}\) An \( a \)-chain of \( T \) is a chain \( H \) such that \( a \in H \) and \( a \) beats all other candidates in \( H \). The set of all \( a \)-chains is denoted as \( H(a, T) \).

For sequential binary voting using majority rule, Banks (1985) completely characterized the equilibrium outcomes for all possible orderings of the candidates when the voters are strategic. The equilibrium set, known as the Banks set, is defined as

\[
BS(T) = \{a \mid \exists \sigma \text{ s.t. } a = S(\mathcal{K}, \sigma, T)\},
\]

where \( S(\cdot , \cdot , \cdot) \) means sophisticated (i.e., strategic) equilibrium.

Given that the Banks set is the most refined set among all other available equilibrium characterizations for binary voting,\(^{23}\) and binary voting is one of the most well-recognized voting rules that attracted researchers’ attention, our characterization of equilibrium of the weakest link game will be stated in reference to this Banks set. For convenience, we will use an alternative definition of the Banks set

\(^{22}\)The notation \( H \) should not be confused with the previously used notation \( \mathcal{H} \) that stands for the set of histories in the weakest link game.

\(^{23}\)See Proposition 6 in Dutta, Jackson and Le Breton (2002).
given in Dutta, Jackson and Le Breton (2002) (see their Proposition 4):

\[
BS(T) = \{ a | \exists H \in H(a, T) \text{ s.t. } \forall b \notin H, \exists c \in H \text{ s.t. } cTb \}.
\]

**Proposition 1** Suppose there are only three voters and an arbitrary but finite number of candidates. Then, under strategic voting, any element of the Banks set can be induced as the winner of the weakest link game for an appropriately chosen tie-breaking rule.

**Proof.** Consider any specific maximal chain \(z_1, \ldots, z_n\), while \(z_{n+1}, \ldots, z_k\) are elements outside this maximal chain.\(^{24}\) So, \(z_1\) belongs to the Banks set. Fix the tie-breaker as follows: \(z_1\) is placed first, followed by any arbitrary but fixed ordering of \(z_2, \ldots, z_n\), which in turn is followed by any arbitrary but fixed ordering of the remaining candidates \(z_{n+1}, \ldots, z_k\).

**Claim 1.** No candidate outside the maximal chain can be the winner of the weakest link game for the specified tie-breaking rule.

Suppose not, so let some \(z_\tau \in \{z_{n+1}, \ldots, z_k\}\) be the winner. By definition of the Banks set, there exists some \(z_i\) in the maximal chain such that \(z_i T z_\tau\). Let \(l, m\) be at least two voters for whom \(z_i \succ_l z_\tau\) and \(z_i \succ_m z_\tau\).

Clearly, \(z_i\) could not have been eliminated in stage \(k - 1\) with \(z_\tau\) becoming the winner: \(l\) and \(m\) each would vote sincerely and \(z_i\) will be the winner instead.

Next consider stage \(k - 2\) when \(z_i\) is eliminated. Suppose the candidates to be voted are: \(z_\tau, z_j, z_i\). The candidate \(z_i\) must have received zero vote, so that \(l\) and \(m\) must have voted for \(z_\tau\) or \(z_j\). If any one of them switches to \(z_i\), then \(z_i\) survives to stage \(k - 1\) and the contestants in stage \(k - 1\) are:

(a) \(z_\tau, z_i\), or
(b) \(z_j, z_i\).

(If \(z_\tau, z_j, z_i\) receive one vote each and \(z_j\) is placed ahead of \(z_\tau\) in the tie-breaker, then \(z_i\) is eliminated. If \(z_j\) receives all three votes, again \(z_\tau\) will be eliminated by the tie-breaker.) If (a), then \(z_i\) goes on to become the winner and the switching will be a dominating strategy. If (b), there are two possibilities:

(b.i) \(z_i T z_j\),
(b.ii) \(z_j T z_i\).

\(^{24}\)A chain is **maximal** if no other set (of candidates) of a higher cardinality will constitute a chain.
Possibility (b.i) again induces the winner $z_i$, making the switch by either $l$ or $m$ to $z_i$ a dominating strategy. Possibility (b.ii) implies that $z_j$ would go on to become the winner, the desirability of which should be examined against the following specification of voter preferences:

- either $z_j \succ_l z_\tau$, or $z_j \succ_m z_\tau$, or both;
- both $z_\tau \succ_l z_j$ and $z_\tau \succ_m z_j$.

The first specification would imply that switching to $z_i$ is a dominating strategy for either $l$ or $m$, if not both. However, if the second specification holds, then neither $l$ nor $m$ would switch their vote to $z_i$. But we argue that the second specification directly contradicts possibility (b.ii) and the fact that $z_i \succ_l z_\tau$ and $z_i \succ_m z_\tau$. To see how, note that

$$
\begin{align*}
z_i & \succ_l z_\tau; \quad z_\tau \succ_l z_j \\
z_i & \succ_m z_\tau; \quad z_\tau \succ_m z_j
\end{align*}
$$

imply the following strict preference orderings:

Voter $l$: $z_i, z_\tau, z_j$

Voter $m$: $z_i, z_\tau, z_j$,

which, in the three-voters scenario, imply that $z_i T z_j$.

Thus, either voter $l$ or $m$ would switch to candidate $z_i$ in stage $k-2$, making $z_\tau$ being the winner an impossibility.

The above analysis for stage $k-2$ is sufficient to establish the claim. Suppose $z_\tau$ goes on to become the winner with $z_i$ eliminated at some stage $j < k-2$, then at least one of the two voters $l$ and $m$ can always continue voting for $z_i$ at least until stage $k-1$ so that $z_i$ is one of the two candidates when stage-$(k-1)$ vote is taken. Then, stage-$(k-1)$ contest is either of category (a) or category (b), in each of which at least one of the two voters $l$ and $m$ can eject $z_\tau$ from being the final winner and strictly gain. Thus, $z_\tau$ can never become the winner, establishing the claim.$^{25}$

**Claim 2.** No candidate within the maximal chain other than the leading element of the chain can become the winner of the weakest link game for the specified tie-breaking rule.

Claim 2 follows immediately, applying the same argument used to prove Claim 1. The key point is, now for any non-leading element of the maximal chain, the

$^{25}$There is no claim here that $z_j$, a likely winner from the switching strategy of $l$ or $m$, is indeed an element of the maximal chain. What is important is that, $z_\tau$ cannot be the winner.
leading element $z_1$ majority dominates, is placed ahead of others in the tie-breaker, and thus serves the role of $z_i$. 

Claims 1 and 2 leave only one possible winner – the leading element of the maximal chain, a member of the Banks set. Q.E.D.

**Proposition 2** Under strategic voting, every element of the Banks set will be the winner of the weakest link game for an appropriately chosen tie-breaking rule.

*Proof.* [The proof requires a fresh approach – Proposition 1 proof won’t work. This is because, in the 3 voters case each voter, singlehandedly, can carry a candidate, barring a possible obstruction from the tie-breaker, all the way up to stage $k - 1$. This won’t be the case for more than 3 voters.]

**Proposition 3** Under strategic voting, the equilibrium outcome of the weakest link game may not belong to the Banks set.

*Proof.* Consider the following strict preference ordering of 4 candidates by 3 voters:

1: $z_2, z_3, z_1, z_4$
2: $z_4, z_1, z_2, z_3$
3: $z_3, z_4, z_1, z_2$

and the tie-breaking rule: $z_2, z_3, z_4, z_1$.

The above preferences give rise to a cycle: $z_1 T z_2 T z_3 T z_4 T z_1$. Also, $z_3 T z_1$, and $z_4 T z_2$. There are only two maximal chains: $z_3, z_4, z_1$; and $z_4, z_1, z_2$. Thus, the Banks set consists of: $z_3, z_4$.

Consider the following strategies.

Stage 1:
1. $z_3$
2. $z_2$
3. $z_4$

Stage 2:
If $z_1$ is eliminated in stage 1 then play $z_2$, the unique outcome by Lemma 2;
If $z_2$ is eliminated in stage 1 then play the ‘Condorcet winner’ $z_3$;
If $z_3$ is eliminated in stage 1 then play the ‘Condorcet winner’ $z_4$;
If $z_4$ is eliminated in stage 1 then play $z_2$, the unique outcome by Lemma 2.\textsuperscript{26}

We now claim that the above strategies will be an equilibrium of the weakest link game and pick $z_2$ as the winner, which is outside the Banks set. As voters choose their proposed stage 1 votes, $z_1$ gets eliminated following which $z_2$ becomes the winner. Next we verify that the proposed strategies will survive the elimination procedure, hence no voter would deviate in stage 1.

Clearly, by deviating voter 1 can do no better than the outcome $z_2$. Moreover, switching from $z_3$ will result in a worse winner in voter 1’s ranking for at least one pair of stage 1 votes by the remaining two voters (not necessarily same as their votes under the proposed strategies), as follows:

\begin{align*}
1. & \quad z_3 \quad z_1 \quad 1. \quad z_3 \quad z_2 \\
2. & \quad z_4 \quad \succ (1) \quad z_4 \quad 2. \quad z_4 \quad \succ (1) \quad z_4 \\
3. & \quad z_2 \quad z_2 \quad 3. \quad z_1 \quad z_1 \\
& \quad (z_2) \quad (z_4) \quad (z_3) \quad (z_4)
\end{align*}

(The winner is indicated in parenthesis underneath the voters’ strategies.) Hence voter 1 will not deviate.

If voter 2 deviates from $z_2$ to $z_3$ or $z_4$, the winner is still $z_2$; if he deviates to $z_1$ then the winner is $z_3$, which is worse for voter 2. Moreover, switching to $z_3$ or $z_4$ will result in a worse winner for voter 2 for at least one pair of stage 1 votes by the remaining two voters, as follows:

\begin{align*}
1. & \quad z_1 \quad z_1 \quad 1. \quad z_1 \quad z_1 \\
2. & \quad z_2 \quad \succ (2) \quad z_3 \quad 2. \quad z_2 \quad \succ (2) \quad z_4 \\
3. & \quad z_4 \quad z_4 \quad 3. \quad z_3 \quad z_3 \\
& \quad (z_4) \quad (z_3) \quad (z_2) \quad (z_3)
\end{align*}

\textsuperscript{26}The strategy of sincere voting in stage 3, which is undominated, is implicit in the description of stage 2 strategies.
Hence voter 2 will not deviate.

If voter 3 deviates from $z_4$ to one of $z_1$, $z_2$ or $z_3$, the winner is still $z_2$ and therefore voter 3 can do no better. Moreover, switching from $z_4$ will result in a worse winner for voter 3 for at least one pair of stage 1 votes by the remaining two voters, as follows:

1. $z_1$  $z_1$
2. $z_3$  $\succ^{(3)}$  $z_3$
3. $z_4$  $z_1$

$(z_3)$  $(z_2)$

1. $z_1$  $z_1$
2. $z_1$  $\succ^{(3)}$  $z_1$
3. $z_4$  $z_3$.

$(z_4)$  $(z_2)$

Hence voter 3 will not deviate. Q.E.D.

4 One-shot voting mechanisms

We now turn our attention to some popular one-shot voting rules to check for Condorcet consistency. This will help to bring the results on sequential voting into sharper focus.

**Proposition 4** Under strategic voting, the plurality rule is not Condorcet consistent.

**Proof.** Consider a 5 voters, 3 candidates scenario, where the voters’ ranking of the candidates are as follows:

1. $z_1, z_2, z_3$
2. $z_3, z_1, z_2$
3. $z_2, z_1, z_3$
4. $z_3, z_1, z_2$
5. $z_1, z_2, z_3$. 

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For this ranking, $z_1$ is the Condorcet winner.

Consider the tie-breaker: $z_2, z_3, z_1$. Under plurality rule an (undominated Nash) equilibrium strategy profile is

$$(z_2, z_3, z_2, z_3, z_2),$$

which picks $z_2$. It is obvious that the proposed strategy profile is a Nash equilibrium. Next we verify that for any voter no other strategy weakly dominates his proposed equilibrium strategy.

First consider the strategy of voter 5. Against the strategy profile

1. $z_1$
2. $z_3$
3. $z_2$
4. $z_3$,

if voter 5 chooses $z_1$ then $z_3$ will be picked, whereas if he chooses $z_2$ then $z_2$ will be picked which is a better alternative for voter 5. Thus, $z_1$ does not weakly dominate $z_2$. Likewise choosing $z_3$ will pick $z_3$, whereas choosing $z_2$ will pick $z_2$. Thus, $z_2$ is undominated for voter 5. By symmetry, $z_2$ is undominated for voter 1 as well.

Now consider the strategy of voter 2. Against the strategy profile

1. $z_1$
3. $z_1$
4. $z_3$
5. $z_2$,

if voter 2 chooses $z_1$ then $z_1$ will be picked, whereas if he chooses $z_3$ then $z_3$ will be picked which is a better alternative for voter 2. Thus, $z_1$ does not weakly dominate $z_3$. Likewise choosing $z_2$ will pick $z_2$, whereas choosing $z_3$ will pick $z_3$ which, for voter 2, is better than $z_2$. Thus, $z_3$ is undominated for voter 2, and by symmetry also for voter 4.

Finally, consider voter 3’s strategy. If he replaces his proposed equilibrium strategy $z_2$ by $z_3$ while the rest of the voters play their proposed equilibrium strategies, $z_3$ will be picked which is worse than $z_2$ for voter 2. Thus, $z_3$ does not weakly dominate $z_2$. To see that $z_1$ does not weakly dominate $z_2$ either, consider the strategy
profile:

1. z₂
2. z₁
4. z₁
5. z₂.

Clearly, z₁ will be picked if voter 3 chooses z₁, whereas voter 3’s preferred alternative z₂ will be picked if he chooses z₂.

Thus, (z₂, z₃, z₂, z₃, z₂) is an equilibrium. Hence, the plurality rule is not Condorcet consistent. Q.E.D.

That the plurality rule is not Condorcet consistent under sincere voting is well-recognized in the literature (see chapter 9 of Moulin, 1988).

In approval voting, each voter partitions the candidates into “equally good” and “equally bad” ones by giving candidates in the first category 1’s and the second category 0’s, and the candidate with a maximal number of votes is elected. See Brahms and Fishburn (1978), or Myerson (2002) for a latest application.

**Proposition 5** Under strategic voting, approval voting is not Condorcet consistent.

**Proof.** Consider a 3 voters, 3 candidates scenario, where the voters’ ranking of the candidates are as follows:

1:  z₁, z₂, z₃
2:  z₃, z₁, z₂
3:  z₂, z₁, z₃.

z₁ is the Condorcet winner. Suppose, as before, the tie-breaker is: z₂, z₃, z₁. The proposed equilibrium strategies under approval voting are as follows:

1. 1, 1, 0
2. 0, 0, 1
3. 0, 1, 0,

where the vote points are respectively for z₁, z₂ and z₃. Denote this strategy profile by σ*. 21
\( \sigma^* \) will pick \( z_2 \). It is obvious that \( \sigma^* \) is a Nash equilibrium. Next we verify that for any voter no other strategy weakly dominates his proposed equilibrium strategy in \( \sigma^* \).

Each voter can choose one of the following 8 strategies:

\[
1, 0, 0; \quad 0, 1, 0; \quad 0, 0, 1; \\
1, 1, 0; \quad 0, 1, 1; \quad 1, 0, 1; \\
1, 1, 1; \quad 0, 0, 0.
\]

First consider the strategy of voter 1. It is easy to see that

\[
\begin{array}{cccc}
1. & 1, 0, 0 & 1, 1, 0 & 1. & 0, 1, 0 & 1, 1, 0 \\
2. & 0, 0, 1 & \prec & 0, 0, 1 & 2. & 1, 0, 0 & \prec & 1, 0, 0 \\
3. & 0, 1, 1 & 0, 1, 1 & 3. & 0, 0, 1 & 0, 0, 1 \\
1. & 0, 0, 1 & 1, 1, 0 & 1. & 0, 1, 1 & 1, 1, 0 \\
2. & 0, 1, 1 & \prec & 0, 1, 1 & 2. & 1, 0, 0 & \prec & 1, 0, 0 \\
3. & 0, 0, 1 & 0, 0, 1 & 3. & 0, 0, 1 & 0, 0, 1 \\
1. & 1, 0, 1 & 1, 1, 0 & 1. & 1, 1, 1 & 1, 1, 0 \\
2. & 0, 0, 1 & \prec & 0, 0, 1 & 2. & 0, 1, 1 & \prec & 0, 1, 0 \\
3. & 1, 0, 0 & 1, 0, 0 & 3. & 0, 0, 1 & 0, 0, 1 \\
1. & 1, 0, 0 & 1, 1, 0 & \\
2. & 1, 0, 0 & \prec & 1, 0, 0 \\
3. & 0, 0, 1 & 0, 0, 1.
\end{array}
\]

For voter 2,

\[
\begin{array}{cccc}
1. & 1, 0, 1 & 1, 0, 1 & 1. & 1, 0, 1 & 1, 0, 1 \\
2. & 1, 0, 0 & \prec & 0, 0, 1 & 2. & 0, 1, 0 & \prec & 0, 0, 1 \\
3. & 1, 0, 0 & 1, 0, 0 & 3. & 0, 1, 0 & 0, 1, 0 \\
1. & 1, 0, 1 & 1, 0, 1 & 1. & 0, 0, 1 & 0, 0, 1 \\
2. & 1, 1, 0 & \prec & 0, 0, 1 & 2. & 0, 1, 1 & \prec & 0, 0, 1 \\
3. & 0, 1, 0 & 0, 1, 0 & 3. & 0, 1, 0 & 0, 1, 0 \\
\end{array}
\]
Finally for voter 3,

1. 1, 0, 1  1, 0, 1
2. 0, 0, 0 ≺ 0, 0, 1
3. 0, 1, 0  0, 1, 0.

Thus, σ* is an equilibrium. Hence, approval voting is not Condorcet consistent.

Q.E.D.

**Proposition 6** Borda voting is not Condorcet consistent, whether the voters are sincere or strategic.
Proof. Consider 7 voters with preferences over 3 candidates as follows:

1: c, a, b  
2: c, a, b  
3: c, a, b  
4: a, b, c  
5: a, b, c  
6: a, c, b  
7: b, c, a.

Consider a general Borda rule with scores $S_2 > S_1 > S_0$, where $S_2$ is the score received by the top candidate and likewise for $S_1$ and $S_0$. Further assume that $S_2 - S_1 = S_1 - S_0$.\footnote{A standard description of Borda voting would assign $S_2 = 2$, $S_1 = 1$, $S_0 = 0$. See chapter 9 of Moulin (1988).} Now, let us first consider sincere voting. Candidate $a$ gets the highest total score which is $3S_1 + 3S_2 + S_0$. Candidate $c$ is a Condorcet winner but he gets less: $3S_2 + 2S_1 + 2S_0$. So Borda rule with sincere voting is not Condorcet consistent.\footnote{Fishburn (1973) made this observation.}

To show that Condorcet consistency will also fail under strategic voting, suppose by way of contradiction that there is a strategy (i.e. a ranking) $R_i$ for voter $i$ that weakly dominates sincere voting. Without loss of generality assume that this voter’s true preference is: $a, b, c$.

We first claim that $b$ cannot placed top in $R_i$. Otherwise, consider the remaining 6 voters and assume that 3 of them submit $a, b, c$ and three submit $b, a, c$. Then clearly sincere voting (which will lead to $a$) is better than $R_i$ (which leads to $b$). Using the same argument, conclude that $c$ cannot be placed top in $R_i$. Hence it must be that $a$ is placed top in $R_i$. Thus if $R_i$ is not sincere voting then it must be: $a, c, b$. But if so, consider the following votes by the remaining 6 voters: 3 vote $c, b, a$ and 3 vote $b, c, a$. Under this profile again sincere voting (leading to $b$) is strictly better for voter $i$ than $R_i$ (leading to $c$).

Q.E.D.

The negative voting rule\footnote{It is also known as the anti-plurality rule (p. 231, Moulin, 1988). See also Myerson (2002).} allows the voters to express only their least desired candidate by giving a point, 0, while giving the remaining candidates all 1’s. The candidate with the highest total points, i.e. the candidate with the fewest 0’s, wins.
Proposition 7  Under strategic voting, the negative voting rule is not Condorcet consistent.

Proof. Consider a 3 voters, 3 candidates scenario, where the voters’ ranking of the candidates are as follows:

1:  \(z_1, z_2, z_3\)
2:  \(z_2, z_3, z_1\)
3:  \(z_1, z_2, z_3\)

\(z_1\) is the Condorcet winner. Suppose the tie-breaker is: \(z_2, z_3, z_1\). The proposed equilibrium strategies under the negative voting rule are as follows:

1. 1,1,0
2. 0,1,1
3. 1,1,0,

where the vote points are respectively for \(z_1, z_2\) and \(z_3\). Denote this strategy profile by \(\sigma^*\).

\(\sigma^*\) will pick \(z_2\). It is obvious that \(\sigma^*\) is a Nash equilibrium. Next we verify that for any voter no other strategy weakly dominates his proposed equilibrium strategy in \(\sigma^*\).

Each voter can choose one of the following 3 strategies:

1,1,0; 1,0,1; 0,1,1.

First consider the strategy of voter 1. It is easy to see that

\[
\begin{align*}
1. & \quad 1,0,1 & 1,0,0 & 1. & \quad 0,1,1 & 1,1,0 \\
2. & \quad 0,1,1 & \prec & 0,1,1 & 2. & \quad 1,0,1 & \prec & 1,0,1 \\
3. & \quad 0,1,1 & 0,1,1 & 3. & \quad 0,1,1 & 0,1,1.
\end{align*}
\]

Symmetrically, the above comparison holds for voter 3 as well.

For voter 2,

\[
\begin{align*}
1. & \quad 1,0,1 & 1,0,1 & 1. & \quad 1,0,1 & 1,0,1 \\
2. & \quad 1,1,0 & \prec & 0,1,1 & 2. & \quad 1,0,1 & \prec & 0,1,1 \\
3. & \quad 1,1,0 & 1,1,0 & 3. & \quad 1,1,0 & 1,1,0.
\end{align*}
\]
Thus, \( \sigma^* \) is an equilibrium. Hence, the negative voting rule is not Condorcet consistent. \( \text{Q.E.D.} \)

Note that the proposed (equilibrium) strategies reflect the voters’ true preferences in the sense that the true least-desired candidates have been assigned 0’s. Thus, Condorcet consistency fails under sincere voting.

5 Conclusion

What initially prompted this work is the adoption of a very simple voting mechanism – the weakest link voting – by a major political party in Great Britain in electing its leader. Surprisingly, the mechanism was not previously examined in the voting or game-theory literature. So, we ask if the mechanism has any especially nice feature that might be lacking in some of the other prominent voting rules. We find the answer to be a positive one: the weakest link voting is Condorcet consistent so long as the voters vote strategically. While Condorcet consistency is not unique to the weakest link voting, it is not widely featured either. This led us to examine a class of sequential voting rules that share the Condorcet consistency property. We also identify some equilibrium properties of the weakest link voting when there is no Condorcet winner.

There are other related aspects of the weakest link voting that might be of interest. An obvious but difficult issue is that of strategic participation by the candidates, similar to the one studied by Dutta, Jackson and Le Breton (2002). In our analysis the set of candidates is exogenously given. In actual selection process strategic participation can alter the eventual winner,\(^{30}\) so whether Condorcet consistency, appropriately defined, holds remains to be seen. The other issue, which is also likely to be complicated, is that of ‘horse trading’. It would require a very different type of modelling – potential candidates try to persuade prospective followers to cast their votes in a particular manner. We leave these and related issues as open problems for future work.

\(^{30}\)Dutta, Jackson and Le Breton (2002) showed this to be the case for binary voting.
Appendix

Lemma 2 Consider the following strict preference ordering of 3 candidates by 3 voters

1: \( z_2, z_3, z_4 \)
2: \( z_3, z_4, z_2 \)
3: \( z_4, z_2, z_3 \),

leading to a cycle: \( z_2 \sim z_3 \sim z_4 \sim z_2 \). For any tie-breaking rule \( z_i, z_j, z_k \), the lowest ranked candidate, \( z_k \), cannot be the winner of the weakest link game. The unique outcome is:

\[
\begin{cases} 
  z_i & \text{if } z_i \sim z_j; \\
  z_j & \text{if } z_j \sim z_i.
\end{cases}
\]

Proof. The candidate placed bottom in the tie-breaker, \( z_k \), must also be the least preferred candidate in one (and only one) of the three voters’ ranking. Denote this voter as voter \( h \).

Given the cycle, either \( z_i \) or \( z_j \) (but not both) must be majority preferred to \( z_k \) and denote this majority winner of \( z_k \) by \( \omega(z_k) \). Voter \( h \), by choosing \( \omega(z_k) \) in stage 1, can ensure that \( z_k \) will not be the winner: \( \omega(z_k) \) will survive stage 1 elimination by the tie-breaking rule; now if \( z_k \) is eliminated in stage 1, our observation holds; alternatively, if \( z_k \) survives stage 1 elimination then it will face \( \omega(z_k) \) in stage 2, in which case \( z_k \) gets eliminated. Therefore, there cannot be an equilibrium of the weakest link game in which \( z_k \) is the winner.

To show that the majority winner between \( z_i \) and \( z_j \) will be the unique winner, consider without loss of generality the two tie-breaking rules that place \( z_2 \) at the bottom:

(a) \( z_4, z_3, z_2 \)

(b) \( z_3, z_4, z_2 \).

The following argument applies equally to tie-breaking rules (a) and (b), except where indicated.

We already established that \( z_2 \) cannot be the winner of the weakest link game. Now voter 1, by choosing \( z_3 \) in stage 1, can always guarantee that either \( z_2 \) or \( z_3 \) will be the winner. Thus, \( z_4 \) cannot be the winner either. Next we argue that \( z_3 \) is the winner.
The equilibrium strategies for selecting $z_3$ will be as follows.

Stage 1:
- Voter 1 chooses $z_3$
- Voter 2 chooses $z_4$
- Voter 3 chooses $z_2$

Stage 2:
- Voters vote sincerely.

Denote the proposed strategies by $\sigma$. With choices restricted to only two candidates, voting sincerely in the second stage is undominated for each voter. Next we check that stage 1 components of $\sigma$ are the Nash best responses.

If voter 1 deviates to $z_2$ or $z_4$, the winner is $z_4$ which is worse for voter 1.
If voter 2 deviates to $z_2$ or $z_3$, the winner is $z_2$ which is worse for voter 2.
If voter 3 deviates to $z_3$ or $z_4$, the winner is unchanged.

Finally we need to check that the strategies, $\sigma$, will survive the elimination procedure. This requires us to check for each voter that switching from one’s proposed stage 1 vote in $\sigma$ to a different vote will result in a worse winner (in the particular voter’s ranking) for at least one pair of stage 1 votes by the remaining two voters (not necessarily same as their $\sigma$ counterparts). This verification is already done above for voters 1 and 2. For voter 3, below we do separate checks for the tie-breaking rules (a) and (b). For the tie-breaking rule (a),

1. $z_2$ $>$ $z_3$
2. $z_2$ $>$ (3) $z_3$
3. $z_2$ $<$ (3) $z_3$

For the tie-breaking rule (b),

1. $z_4$ $>$ $z_3$
2. $z_4$ $>$ (3) $z_3$
3. $z_2$ $<$ (3) $z_4$

To see how the strict domination works in the above comparisons, apply sincere voting in stage 2 to the candidates who survive stage 1 voting. This completes the proof that $\sigma$ will pick $z_3$ as the unique winner.

Q.E.D.
Lemma 3  For the following preference ordering of 4 candidates by 3 voters

1:  \( z_2, z_1, z_3, z_4 \)
2:  \( z_3, z_1, z_4, z_2 \)
3:  \( z_4, z_2, z_1, z_3 \)

leading to a cycle and the tie-breaking rule \( z_1, z_4, z_3, z_2 \), the element \( z_1 \) which is a member of the Banks set will be the winner of the weakest link game.

Proof. The cycle is: \( z_4Tz_2Tz_3Tz_4 \). Of this cycle, \( z_1Tz_3Tz_4 \) is a maximal chain and \( z_2 \) is outside this chain; \( z_2Tz_1 \) and \( z_2Tz_3 \) but \( z_4Tz_2 \). So \( z_1 \) is an element of the Banks set. In the following we will argue that \( z_1 \) will emerge as the winner of the weakest link game for the given tie-breaking rule.

Consider the following strategies.

Stage 1 strategies \( \sigma_1 \) are:
- Voter 1 chooses \( z_1 \)
- Voter 2 chooses \( z_4 \)
- Voter 3 chooses \( z_3 \);

Stage 2 strategies \( \sigma_2 \) are:
- If \( z_2 \) is eliminated in stage 1 then play the ‘Condorcet winner’ \( z_1 \);
- If \( z_1 \) is eliminated in stage 1 then play \( z_3 \), the unique outcome by Lemma 2;
- If \( z_4 \) is eliminated in stage 1 then play the ‘Condorcet winner’ \( z_2 \);
- If \( z_3 \) is eliminated in stage 1 then play \( z_1 \), the unique outcome by Lemma 2.

Denote \( \sigma = \sigma_1 \times \sigma_2 \).\(^{31}\)

We now verify that \( \sigma \) will pick \( z_1 \) as the winner. First, as voters choose their stage 1 votes specified by \( \sigma_1 \), \( z_2 \) gets eliminated following which \( z_1 \) becomes the winner by Theorem 1 (as \( z_1 \) is the Condorcet winner when restricted to only candidates \( z_1, z_3, z_4 \)). Next check that in stage 1 no voter would like to deviate from his \( \sigma_1 \) strategy.

If voter 1 deviates from \( z_1 \) to \( z_2 \), the outcome is \( z_3 \) which is worse for voter 1. On the other hand, if he deviates to \( z_3 \) or \( z_4 \), the outcome remains unchanged at \( z_1 \). Furthermore, switching to \( z_3 \) or \( z_4 \) will result in a worse winner for voter 1 for

\(^{31}\)Sincere voting in stage 3, which is undominated, is part of the description of stage 2 strategies and hence not explicitly noted in \( \sigma \).
at least one pair of stage 1 votes by voters 2 and 3, as follows:

1. $z_1$ $z_3$
2. $z_2$ $\succ_{(1)} z_2$
3. $z_4$ $z_4$

1. $z_1$ $z_4$
2. $z_3$ $\succ_{(1)} z_3$
3. $z_2$ $z_2$

In the first comparison above, the vote of $z_1$ by voter 1 induces the winner $z_1$, whereas a vote of $z_3$ induces the winner $z_3$. In the second comparison the vote of $z_1$ induces the winner $z_2$, whereas a vote of $z_4$ induces the winner $z_3$. Hence voter 1 will not deviate.

If voter 2 deviates and votes for $z_2$ then $z_2$ becomes the winner, which is worse for voter 2 and thus voter 2 will not deviate to $z_2$. On the other hand, if he deviates to $z_1$ or $z_3$, by the tie-breaking rule $z_2$ will be eliminated resulting in $z_1$ as the winner. Moreover, switching to $z_1$ or $z_3$ will result in a worse winner for voter 2 for at least one pair of stage 1 votes by voters 1 and 3, as follows:

1. $z_3$ $z_3$
2. $z_4$ $\succ_{(2)} z_1$
3. $z_2$ $z_2$

1. $z_3$ $z_3$
2. $z_4$ $\succ_{(2)} z_3$
3. $z_2$ $z_2$

In the above comparisons, the vote of $z_4$ by voter 2 induces the winner $z_3$, whereas a vote of $z_1$ or $z_3$ induces the winner $z_2$. Hence voter 2 will not deviate to $z_1$ or $z_3$ either.

If voter 3 deviates from $z_3$ to vote for $z_1$, $z_2$ or $z_4$, the winner is $z_1$. However, switching to any of $z_1$, $z_2$ or $z_4$ will result in a worse winner for voter 3 for at least one pair of stage 1 votes by voters 1 and 2, as follows:

1. $z_1$ $z_1$
2. $z_2$ $\succ_{(3)} z_2$
3. $z_3$ $z_1$

1. $z_1$ $z_1$
2. $z_2$ $\succ_{(3)} z_2$
3. $z_3$ $z_2$

1. $z_2$
2. $z_3$ $\succ_{(3)} z_3$
3. $z_3$ $z_4$.
In the above comparisons, the vote of $z_3$ by voter 3 induced the winner $z_2$; in contrast, switching to a vote of $z_1$ or $z_2$ will induce the winner $z_1$ and switching to a vote of $z_4$ will induce the winner $z_3$, and both these alternative winners $z_1$ and $z_3$ are worse relative to $z_2$ in voter 3’s ranking. So voter 3 will not deviate from $z_3$. \[ \text{Q.E.D.} \]

References


