Cross-Owned Firms Competing in Auctions
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Abstract

The paper studies the effect of ownership links among bidders in auctions. Firstly, it is shown that in first-price, second-price, and all-pay auctions, ownership links damage both the seller and society; the bidders too may be impaired by the seller’s strategic reaction. Secondly, the optimal selling procedure is characterized: in sharp contrast with standard auctions, both the seller and society gain from ownership links. In the last part of the paper the analysis is extended to the case of strategic entry.

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1 Introduction

Does cross holding among competing firms hurt society? While a firm’s takeover of a competitor raises obvious concern about adverse effects on competition, several economists have expressed worry also about partial ownership, and even passive investment when neither anticompetitive intention nor tacit collusion exists.\(^2\) For instance, Reynolds and Snapp (1986) show that ownership links lessen competition in non-collusive oligopolistic markets and, as a consequence, bring about both an increase in the equilibrium price and a reduction in the output level (see also Bresnahan and Salop, 1986). American antitrust authorities, on the other hand, have challenged ownership arrangements, under the Sherman Act if there was evidence of anticompetitive intent, and under the Clayton Act if they conveyed control or accounted for a large share of industry output. In establishing their tolerant approach, American agencies have probably considered both the pros and cons of lessening competition – in particular the positive effects on R&D and technology transfers (for more on this, see Reynolds and Snapp, 1986). However, thorough examination of sentences reveals that they seem to ignore the importance of a firm’s strategic motivation for passive investments, namely that a commitment to a less competitive behavior, which is implicit in the investment decision, might induce rivals to behave less competitively themselves; hence the claim that antitrust agencies should be less tolerant on passive ownership links between competitors (Gilo, 2000).

This paper focuses on passive ownership links among bidders in auctions. In the literature, property linkages between the seller and a bidder have been considered in Bulow et al. (1999), where it is shown that bidding becomes more aggressive. However, to the best of my knowledge, ownership links among bidders have not been examined. The legal practice for auctions and for public procurement in the U.S. and the E.U. makes no specific provision about passive ownership. One exception is the Italian law for public procurement, which forbids the simultaneous participation in the same auction of both a controlled corporation and its controlling one; double participation is also forbidden, in that a member of a consortium that participates in the auction is not allowed to participate individually. The rationale may be to prevent collusive behavior (De Fraja, 2002). All in all, passive ownership links among bidders does not seem to have attracted the attention of economists or legislators.

The impact of cross holding on the outcome of an auction is not obvious. Two opposite intuitions come to mind. The first one is that by decreasing its bid, a firm may induce rivals to do the same, with a positive impact on the winner’s surplus, an adverse impact on the seller’s revenue, and perhaps an adverse impact on social surplus if the seller’s best response is to increase his reservation price. Another intuition, however, points out that bidding firms do have an interest in the mere fact that the object is sold, because losing firms will share in the winner’s surplus. A clever seller could find a way to take advantage of this.

\(^2\)Passive investments are those in which investors do not seek to gain influence over the competitor’s activities or to access the competitor’s sensitive information (see Gilo, 2000, for an extensive discussion of passive investments).
of this by extracting some of the surplus from losing bidders; in this manner, perhaps he could gain, rather than lose from the presence of ownership links. In this paper, we show that the optimal auction in the presence of ownership links among bidders exploits the second intuition. The seller extracts part of each firm’s (direct and indirect) surplus by committing not to run the auction unless each potential bidder pays for the auction to be held. For concreteness, we shall refer to the following set up. The seller of an indivisible object faces \( N \geq 2 \) buyers and \( M \geq N \) holding companies. Each holding company controls at most one buyer and owns minority shares in the remaining ones; each buyer’s decisions are made in the sole interest of the controlling holding; the seller has no proprietary links with any buyers. This set up captures important aspects of proprietary links in the real world. For instance, in Italy, the most common form of company control is through pyramidal groups headed by a holding company at the top (Brunello et al. 2001); pyramidal groups are also very frequent in France and Belgium (see Becht and Roell, 1999, who compare Continental Europe with the U.K. and the U.S.). Holding companies are very important also in East Asia: in 1980, in Japan 65 of the 100 largest firms belonged to the 16 largest holding companies, which controlled 26% of the capital of all non financial companies (Hamilton et al., 1990).

The paper is organized as follows. Section 2 deals with one-stage auctions; since entry decisions and bids are simultaneous, we can safely assume that each potential bidder participates in the auction. Firstly, the equilibrium of the first-price sealed-bid auction is characterized; then, the analysis is extended to the second-price and the all-pay auctions; finally, the revenue-maximizing selling procedure is studied. Section 3 deals with two-stage auctions in which bids are made after each player has observed the entry decisions of the others; in this case firms face a trade-off between entering and staying away. Section 4 summarizes the results and concludes.

## 2 The one-stage model

The following is common knowledge. A risk-neutral, revenue maximizing seller has a single, indivisible object to sell. His valuation of the object is zero. There are \( N \) risk-neutral firms indexed by \( i \in I = \{1, \ldots, N\} \). Each firm’s possible valuation of the object, \( v \), is continuously distributed and drawn independently from a common distribution, \( F \), which is strictly increasing and at least twice continuously differentiable on the interval \( T = [0, v^+] \), with \( F(0) = 0 \) and \( F(v^+) = 1 \).

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3The situation is reminiscent of Jehiel et al. (1996), where the players that would be damaged by the sale of an object pay the seller to renounce selling it. However, both in that paper and a later one by the same authors (Jehiel et al., 1999), which extends the analysis to auctions with negative externalities, it is assumed that external effects are of a given size; instead, in our framework, they are endogenously determined.

4Interestingly, the control premium for quoted stocks is very high in Italy: Zingales (1994) estimates that the private benefit of control is worth more than 60 percent of the value of nonvoting equity. See also Bianco and Casavola, 1999, who give empirical evidence on the limited protection of minority shareholders in Italy.
The probability density for \( v \) on \( T \) is \( f = F' \). The following standard regularity condition holds: \( \frac{f(v)}{f(v_0)} \) is strictly increasing with \( v \). Each firm maximizes its total expected payoff, which is a weighted sum of the surpluses of all firms: the non-negative weight firm \( i \) gives to the surplus of firm \( k \) is denoted by \( \alpha_{ik} \), \( k = 1, ..., N \). It is assumed that \( \alpha_{ii} > \sum_{k \neq i} \alpha_{ik}, i, k \in I \), and \( \sum_{k \neq i} \alpha_{ik} = \lambda \) for all \( i \), so bidders’ symmetry obtains, as will be clear in a moment. We will call \( \lambda \in [0,1] \) the concern parameter. Finally, we assume that the bidders do not collude.

The model naturally extends the benchmark auction model (see e.g. McAfee and McMillan, 1987) to the case of ownership links. The structure of weights mirrors the property structure we discussed in the introduction. For example, let \( N = 2, M = 3 \) and suppose the bidding in control of firm 1 owns a 40% share in this firm and a 20% share in firm 2, the holding in control of firm 2 owns a 60% share in this firm, and a 30% share in firm 1, and the third holding owns the remaining shares, namely 30% of firm 1 and 20% of firm 2. Then it is \( \alpha_{11} = .4, \alpha_{12} = .2, \alpha_{21} = .3, \alpha_{22} = .6 \), and therefore \( \lambda = 1/2 \).

Having described our general setting, we are ready to examine some specific selling procedures.

### 2.1 The first-price sealed-bid auction

In this auction, the seller openly announces a reserve price \( b_\ast \in [0, v^+) \); next, bidder \( i \) submits a sealed bid \( b_i \in [0, v_i] \) \( i \in I \); finally, if the highest bid is greater than the reserve price, the highest bidder is awarded the object at a price equal to its offer — otherwise the seller keeps the object. The losers pay nothing.

Nash equilibria of this auction could be characterized by standard methods; however, it is illuminating to resort to the revelation mechanism approach. In this approach, first the seller announces \( p_i = p_i(v^d) : T^N \rightarrow R_+ \), and \( x_i = x_i(v^d) : T^N \rightarrow R_+, i \in I \), which are, respectively, bidder \( i \)’s probability of getting the object and its payment to the seller when \( v^d \) is the vector of reported types, that is the vector of reported valuations. Then, the bidders simultaneously and confidentially report their valuations to the seller. Finally, payments are made and the good is assigned through a lottery with the announced probabilities. Thanks to the revelation principle we can restrict our attention to the truth-telling equilibria. Let \( f(v) = \prod_j f(v_j), f(v) = \prod_j df_j, v_{-i} = (v_1, v_2, ..., v_{i-1}, v_{i+1}, ..., v_n) \in T_{-i}, (v^d_i, v_{-i}) = (v_1, v_2, ..., v_{i-1}, v^d_i, v_{i+1}, ..., v_n), f(v_{-i}) = \prod_{j \neq i} f(v_j), \) and \( df_{-i} = \prod_{j \neq i} dv_j \). Bidder \( i \)’s total payoff, when it is of type \( v_i \) and reports \( v^d_i \) while the other bidders are telling the truth, is \( U_i (v_i, v^d_i) = -\frac{1}{\lambda} \).

\(^5\)A similar structure applies to the case in which firms have direct ownership links. For instance, let \( N = 2 \) and suppose each firm owns 60% of its equity and 40% of the other firm. Then, letting \( S_i^p \) and \( U_i \) denote firm \( i \)’s expected own surplus and overall surplus respectively, we have: \( U_1 = \frac{20}{24} S_1^p + \frac{10}{24} U_2 \) and \( U_2 = \frac{20}{24} S_2^p + \frac{10}{24} U_1 \); hence \( U_1 = \alpha_{11} S_1^p + \alpha_{12} S_2^p \) implies \( \alpha_{11} = \frac{20}{24}, \alpha_{12} = \frac{10}{24} \), and analogously \( U_2 = \alpha_{22} S_2^p + \alpha_{21} S_1^p \) implies \( \alpha_{22} = \frac{20}{24}, \alpha_{21} = \frac{10}{24} \); finally \( \lambda = \frac{1}{4} \).
\[ \int_{T^{-1}} \sum_k \alpha_{ik} \left[ v_k p_k (v_i^0, v_{-i}) - x_k (v_i^0, v_{-i}) \right] f (v_{-i}) \mathrm{d}v_{-i}. \]

The seller’s expected revenue when all bidders tell the truth is

\[ U_s = \int_{T^{-1}} \sum_{k=1}^N x_k (v) f (v) \mathrm{d}v \]

In searching for an optimal solution, we only consider policy functions that are piecewise continuously differentiable. Let \( Q_i (v^0_i) = \int_{T_{-i}} p_i (v_i^0, v_{-i}) f (v_{-i}) \mathrm{d}v_{-i} \)

\( v_i \), and \( v_i^0 \).

\( R = \) Recalling the definition of \( Q_i (v_i^0) \) allows us to write

\[ \int_{T_{-i}} \sum_k \alpha_{ik} \left[ v_k p_k (v_i, v_{-i}) - x_k (v_i, v_{-i}) \right] - \alpha_{ii} \int_{T_{-i}} p_i (s_i, v_{-i}) \mathrm{d}si \] \[ = U_i (v_i (0, 0), v_i \in T, i \in I) \]

The rules of the first-price sealed-bid auction translate into the following additional restrictions on the direct revelation mechanism. For \( v \in T^N, i \in I \)

\[ R_1 : p_i (v) = \begin{cases} 1, & \text{if } B_i (v_i, b_s) > \text{Max} (b_s, B_i (v_j, b_s) \text{all } j \neq i) \\ 0 & \text{otherwise} \end{cases} \]

\[ R_2 : x_i (v) = \begin{cases} B_i (v_i, b_s), & \text{if } p_i (v) = 1 \\ 0, & \text{otherwise} \end{cases} \]

where \( B_i (v_i, b_s) \) is bidder \( i \)'s optimal bid when it is of type \( v_i \) and the reserve price is \( b_s \).

Now bidder \( i \)'s action set is the interval \([0, v_i]\) of possible reports, the seller’s action set is made up of the interval of possible reserve prices and the space of functions \( B_i = B_i (v_i, b_s) : [0, v_i] \times T \rightarrow [0, v_i], i \in I \). Thanks to our assumptions, we can focus on the class of bidding functions that are monotonically strictly increasing with the bidder’s type (later we will confirm that this class of solutions is not empty).

The principal solves:

\[ \text{Max}_{B_i} \quad U_s \]

s.t. (1), \( R_1, R_2, \sum_{k=1}^N p_k (v) \leq 1, p_k (v) \geq 0, v \in T^N, k \in I \).

Using \( R_1 \) and \( R_2 \), we can write (1), for \( v_i \geq b_s \) and any given \( v_i \), as

\[ \alpha_{ii} \left[ (v_i - B_i (v_i)) \prod_{j \neq i} F (B_j^{-1} (B_i (v_i))) - \int_{T_{-i}} \prod_{j \neq i} F (B_j^{-1} (B_i (v_i))) \mathrm{d}v_{-i} \right] + \]

\[ \sum_{k \neq i} \alpha_{ik} \int_{v_i}^{v_i^+} (v_i - B_i (v_i)) \prod_{j \neq i, j \neq k} F (B_j^{-1} (B_i (v_i))) f (v_i) \mathrm{d}v_i = \]

\[ \sum_{k \neq i} \alpha_{ik} \int_{v_i}^{v_i^+} (v_i - B_i (v_i)) \prod_{j \neq i, j \neq k} F (B_j^{-1} (B_i (v_i))) f (v_i) \mathrm{d}v_i, i \in I \] (3)

(Here I am simplifying notation, omitting the reserve price when confusion does not arise.) The left side of (3) is \( U_i (v_i, v_i) - \alpha_{ii} \int_{T_{-i}} Q_i (s_i) \mathrm{d}si, \) the right side is \( U_i (0, 0) \).

We will show that a symmetric solution exists and is unique in the class of symmetric solutions. Let \( B_i (v_i) = B (v_i), \frac{\partial B_i}{\partial v_i} > 0, v_i \in (b_s, v^+), i \in I \).

Straightforward manipulations of (3) yield

\[ B (v_i) = v_i - \frac{1}{F (v_i)^{v_i^+}} \times \]
\[
\left( \int_{b_s}^{v_i} F^{N-1}(z) \, dz + \lambda \int_{b_s}^{v_i} (z - B(z)) F^{N-2}(z) f(z) \, dz \right), \, b_s \leq v_i \leq v^+\quad (4).
\]

We can obtain a closed form for the bidding function as follow. Differentiating (4) with respect to \(v_i\), we get \(B'(v_i) = (N - 1 - \lambda) (v_i - B(v_i)) f(v_i) / F(v_i)\). This linear differential equation, together with the condition \(B(b_s) = b_s\), yields the bidding function

\[
B(v_i) = v_i - \frac{1}{\lambda - 1} \int_{b_s}^{v_i} F^{N-1}(v) \, dv, \, b_s \leq v_i \leq v^+\quad (5).
\]

\(B\) has the properties we assumed above: it is continuously differentiable and strictly increasing with the bidder’s valuation.\(^6\) Notice also that \(B\) can be seen as a standard bidding function, in which the distribution of types, \(F\), has replaced with the distribution \(G_\delta = F^{1-\delta}\), where \(\delta = \lambda/(N - 1)\). \(G_\delta\) is such that \(G_{\delta'}\) first order stochastically dominates \(G_{\delta''}\) whenever \(\delta' < \delta''\). This means that, for given \(b_s\), the bid of a firm of a given valuation is decreasing with the intensity of ownership links.

Finally, the equilibrium bidding function \(B^e\) is such that \(B^e(v_i) = B(v_i, b^*_s)\), where \(b^*_s\) maximizes \(U^e_s\). Now we are ready to prove the following.

**Proposition 1.** In the equilibrium of the first-price sealed-bid auction:

a) If the object is sold, it is assigned to the bidder with the highest valuation.

b) The optimal reserve price is higher than the seller’s valuation, hence the selling procedure is not ex post efficient.

c) Both the seller’s revenue and the social surplus are strictly decreasing with the concern parameter \(\lambda\).

**Proof:** See Appendix.

**Remark.** In equilibrium, the ex ante expected surplus to a bidder may be either decreasing or increasing with the concern parameter \(\lambda\) (see appendix).

The intuition behind these results is as follows. Assume \(\lambda = 0\) initially, and then let \(\lambda\) increase by a small amount. Suppose that both the seller’s reserve price and bids remain fixed at the values which are optimal at \(\lambda = 0\). A bidder’s expected marginal gain from increasing its bid, which previously was zero, now becomes negative, since winning the auction implies losing the “external” surplus deriving from ownership links. Therefore, bids decrease. The only instrument the seller has to oppose this effect is to increase the reserve price, which, however, raises the probability of not selling the object and causes the social surplus to decrease. This line of reasoning can be repeated for any initial value of \(\lambda\). The seller’s revenue decreases because the effect of bid reduction is first order while the increase in reserve price is only second order. Instead, bidder \(i\)’s ex ante expected surplus, \(\int_{b_i}^{1} [v_i - B^e(v_i)] F^{N-1}(v_i) f(v_i) \, dv_i\), is affected by two first order effects of opposite sign.

**Example 1.** There are two bidders, and valuations are uniformly distributed in the unit interval. Under these assumptions, the bidding function is \(B(v_i) = v_i - \frac{1}{\lambda - 1} \int_{b_s}^{v_i} v^{1-\lambda} \, dv, \, b_s \leq v_i \leq v^+, \, i = 1, 2\). The first order condition for maximization of the seller’s objective function, \(2 \int_{b_s}^{1} B(v_i) v_i \, dv_i\), is

\[\text{For } 0 \leq v < b_s, \, B(v) \text{ is not determined, but this is irrelevant as, by assumption, } B(v) \leq v \text{ for all } v.\]
decreasing its bid, since such a deviation is without effect on its marginal surplus, in the same manner as in Section 2.1 we can write (2) for values which are optimal at $P_i$ and then let $\frac{\partial}{\partial F}$ with $v_i$. Also holds for the second-price and the all-pay auctions.

2.2 Other selling procedures: the second-price sealed-bid and the all-pay auctions

Are the above results particular to the first-price sealed-bid auction or are they common to other auctions? In this section we shall verify that Proposition 1 also holds for the second-price and the all-pay auctions.

The second-price sealed-bid auction

In this auction the winner, if any, is the highest bidder and it pays either the second highest bid or the seller’s reserve price, whichever is higher. Proceeding in the same manner as in Section 2.1 we can write (2) for $b_k \leq v_i \leq v^+$, as $v_i F(v_i) N^{-1} - b_k F(b_k) N^{-1} - \int_{b_k}^{v_i} F(u) N^{-1}du - \sum_{k \neq i} \int_{b_k}^{v_i} B(v_k) F(v_k) N^{-2} f(v_k)dv_k + \sum_{k \neq i} \frac{\partial}{\partial F} \int_{v_i}^{v^+} (v_k F(v_k) N^{-2} - B(v_k) F(v_k) N^{-2}) f(v_k)dv_k - \sum_{k \neq i} \frac{\partial}{\partial F} \int_{b_k}^{v_i} (1 - F(u)) B(u)(N - 2) F(u)_{\max}(0, N - 3) f(u)du = \sum_{k \neq i} \frac{\partial}{\partial F} \int_{b_k}^{v_i} f(v_k) N^{-2} - b_k F(b_k) N^{-2} f(v_k)dv_k - \sum_{k \neq i} \frac{\partial}{\partial F} \int_{b_k}^{v_i} (1 - F(u)) B(u)(N - 2) F(u)_{\max}(0, N - 3) f(u)du.

B(v_i) = \frac{1}{\lambda F(v_i) N^{-2}(1 - F(v_i))} \times \left[ (N - 1) \int_{b_i}^{v_i} (v - B(v)) F(v) N^{-2} f(v) dv - \lambda \int_{b_i}^{v_i} v F(v) N^{-2} f(v) dv + \lambda b_i F(b_i) N^{-2} (1 - F(b_i)) + \lambda \int_{b_i}^{v_i} (1 - F(u)) B(u)(N - 2) F(u)_{\max}(0, N - 3) f(u) du \right].

We can obtain a closed form for the bidding function by differentiating (6) with respect to $v_i$. It results $B'(v_i) = \frac{(N - 1)(1 - \lambda)}{\lambda} \left( v_i - B(v_i) \right) \frac{f(v_i)}{1 - F(v_i)}$. Integrating yields

\[ B(v_i) = \left( 1 - F(v_i) \right)^{(N - 1)(1 - \lambda)} \left( \frac{1 - \lambda}{\lambda} \right) G(v_i) + C_1 \]

where $G(v_i)$ is an antiderivative of $\frac{f(v_i)}{1 - F(v_i)} (1 - F(v_i))^{-\frac{(N - 1)(1 - \lambda)}{\lambda}}$, and $C_1$ is such that $B(b_i, b_s) = b_s$.

In the appendix it is shown that in equilibrium the seller’s expected revenue and the reserve price are, respectively, strictly decreasing and strictly increasing with $\lambda$.

The intuition behind this result is the following. Assume $\lambda = 0$ initially, and then let $\lambda$ increase by a small amount. Suppose bids remain fixed at the values which are optimal at $\lambda = 0$. Firm $i$ perceives that it gains from marginally decreasing its bid, since such a deviation is without effect on its marginal surplus,
which is zero, but increases the expected surplus of the other firms, which pays
less when they win.\footnote{Suppose \( N = 2 \). When firm 2 bids its true valuation while firm 1 of valuation \( v_1 \) and bids
\( b_1 \), firm 1’s expected payoff is:
\[
U_1(b_1) = \int_{v_1}^{b_1} (v_1 - v_2) f(v_2)dv_2 + \lambda \int_{v_1}^{b_1} (v_2 - b_1) f(v_2)dv_2.
\]
Now, it is\[\frac{dU_1}{dv_1} = (v_1 - b_1) f(b_1) - \lambda [1 - F(b_1)].\]When \( b_1 = v_1 \) this derivative is strictly
negative for \( v_1 < v^+ \).

Example 2. Let \( N = 2 \), \( F(v) = v, v^+ = 1 \). Then \( B(v_i, b_s) = \frac{\lambda - v_i + \lambda}{2(1 - b_s)} \frac{2\lambda - 1}{\lambda - 1} \) for \( \lambda \neq \frac{1}{2} \), and \( B(v_i, b_s) = -\ln (1 - v_i) \ln (1 - b_s) \).

\( v_i \), for \( \lambda = 1/2 \). \( b^*_s \) is a maximizer of \( U_s = 2 \int_{b_s}^1 B(v_2, b_s)(1 - F(v_2)) f(v_2)dv_2 \)
+ \( 2(1 - b_s)b^*_s \). It results \( b^*_s = \frac{1 - \lambda + \sqrt{(2\lambda - 1)(1 - b_s) + 1}}{2(1 - \lambda)} \) for all \( \lambda \). It is immediately seen that \( b^*_s \) is increasing with \( \lambda \), while both the seller’s expected revenue and a bidder’s ex ante expected surplus are decreasing with \( \lambda \).

The all-pay auction

This auction only differs from the first-price auction in that each bidder, and not the winner only, pays what it offers. Proceeding as in Section 2.1 we can write (2), for \( b_s \leq v_i \leq v^+, i \in I \), as
\[
v_i F^{N-1}(v_i) - B(v_i) - \int_{v_i}^{v^+} F^{N-1}(v)dv + \sum_{k \geq i} \alpha_k \int_{v_i}^{v^+} v_k F^{N-2}(v_k) f(v_k) dv_k - \int_{v_i}^{v^+} B(v_k) f(v_k) dv_k = \sum_{k \geq i} \alpha_k \int_{v_i}^{v^+} v_k F^{N-2}(v_k) f(v_k) dv_k - \int_{v_i}^{v^+} B(v_k) f(v_k) dv_k,\]
where \( v^\circ \) is the valuation that renders bidder \( i \) indifferent between offering the seller’s reserve price and zero, and therefore \( v^\circ F^{N-1}(v^\circ) = b_s \). From this expression we get
\[
B(v_i) = \begin{cases} 
v_i F^{N-1}(v_i) - \int_{v_i}^{v^+} F^{N-1}(v)dv - \lambda \int_{v_i}^{v^+} v F^{N-2}(v) f(v) dv, & v^\circ \leq v_i \leq v^+, \\
0, & i f v_i < v^\circ.
\end{cases}
\]
(8).

Simple calculations confirm that the seller’s expected revenue in equilibrium is strictly decreasing with \( \lambda \), while the optimal reserve price is strictly increasing with \( \lambda \).

Example 3. Let \( N = 2 \), \( F(v) = v, v^+ = 1 \). Then \( B(v, b_s) = \frac{1}{2} v^2(1 - \lambda) + \frac{1}{2} (v^\circ)^2(1 + \lambda) \). Furthermore, the optimal value of \( b_s \) is \( b^*_s = \frac{(1 + \lambda)^2}{2(2 + \lambda)^2} \), and hence
\[
B(v) = \frac{1}{2} v^2(1 - \lambda) + \frac{1}{2(2 + \lambda)^2} (1 + \lambda).
\]
Notice that in deriving the bidding functions we did not impose any condition on \( U_1(0, 0) \) – of course it is non negative, since no bidder’s own expected surplus can be negative in an equilibrium of a standard auction. Therefore, one cannot expect that the revenue equivalence theorem holds in our framework – recall that a necessary condition for the theorem is that in equilibrium \( U_i(0, 0) \) be the same for any auction procedure. Indeed, it is easily seen that, under the assumptions of the examples, for any value of \( \lambda \), the first-price auction is more profitable than the second-price auction, which on its turn is better than the all-pay auction (see
be written...

Finally, recalling the definition of... follows, after an integration,

\[ f(v) \]

Thanks to symmetry, from (2) we get

\[ (4.9) \text{ in Myerson}(1981): \]

The concern parameter

In Myerson’s (1981).

Since standard auctions are not equivalent, the question arises of what is the optimal selling procedure.

2.3 The revenue-maximizing selling procedure

The formal characterization of the best procedure is not difficult. The seller’s problem is \( \max_{(p, x)} U_s \), s.t. : truth-telling constraint (1), \( \sum_{k=1}^{N} p_k(v) \leq 1 \), \( p_i(v) \geq 0 \), \( v \in T^N \), \( U_i(v_i, v_j) \geq 0 \), \( v_i \in T \), \( i \in I \), which has the same format as Myerson’s (1981).

Notice that the individual rationality constraint requires the total payoff \( U(v_i, v_j) \) to be non-negative for all \( v_i \): this is a crucial difference with respect to standard auctions, in which, as we know, it is required that a bidder’s expected own surplus be non-negative.

The following proposition contains the main results.

\[ \textit{Proposition 3.} \] The optimal selling procedure is such that:

a) If the object is sold, it is assigned to the bidder with the highest valuation.

b) The payoff to the bidder with the lowest possible valuation is zero.

c) The procedure is not ex post efficient.

d) Both the seller’s revenue and social surplus are strictly increasing with the concern parameter \( \lambda \).

Proof. By simple manipulations, we can put \( U_s \) in a form that parallels equation (4.9) in Myerson (1981): \( U_s = \int_{T^N} \sum_{k \in I} (v_k p_k(v) - [v_k p_k(v) - x_k(v)]) f(v) dv \).

Thanks to symmetry, from (2) we get \( U_i(0, 0) = \int_{T^N} \sum_{k \in I} a_{ik} (v_k p_k(v) - x_k(v)) \times f(v) dv - \alpha_{ii} \int_{0}^{v_i} \int_{0}^{v_i} Q_i(s_i) ds_i f(v_i) dv_i \) for all \( v_i \in T \), \( i \in I \). From this it follows, after an integration, \( \alpha_{ii} (1 + \lambda) \int_{T^N} [v_k p_k(v) - x_k(v)] f(v) dv = \alpha_{ii} \int_{0}^{v_i} Q_i(v_i) [1 - F(v_i)] dv_i + U_i(0, 0), \quad i \in I \).

Finally, recalling the definition of \( Q_i(v_i) \) allows the seller’s expected revenue to be written

\[ U_s = -\sum_{i \in I} \frac{U_i(0, 0)}{\alpha_{ii}(1+\lambda)} + \int_{T^N} \sum_{i \in I} [v_i - \frac{1}{1+\lambda} \frac{1-F(v_i)}{f(v_i)}] p_i(v) f(v) dv \]

From (9) we see that it is in the seller’s interest to make \( U_i(0, 0) \) equal to zero for all \( i \), and to set \( p_i(v_i, v_{-i}) \) equal to one (zero) whenever \( c(v_i) = v_i - \frac{1}{1+\lambda} \frac{1-F(v_i)}{f(v_i)} \).
is positive (non-positive). Under our assumptions, $c(v_i)$ is a monotone strictly increasing function of $v_i$. For any vector $v_{-i}$ consider $z(v_{-i}) = \inf\{s : c(s) \geq 0$ and $c(s) \geq c(v_j)$ for all $j \neq i\}$. Then, in equilibrium
\[
p_i(s_i, v_{-i})= \begin{cases} 
1 & \text{if } s_i \geq z(v_{-i}) \\
0 & \text{if } s_i < z(v_{-i})
\end{cases}, \quad \text{and}
\]
\[
\int_0^{v_i} p_i(s_i, v_{-i}) \, ds_i = \begin{cases} 
v_i - z(v_{-i}) & \text{if } v_i \geq z(v_{-i}) \\
0 & \text{if } v_i < z(v_{-i})
\end{cases}.
\]
$p_i(v) \in I \cup T^0$ is piecewise continuously differentiable, as assumed above; it depends positively upon $\lambda$ via the condition $c(s_i) \geq 0$. Since the value $s^*$ that solves $c(s_i) = 0$ is a strictly decreasing function of $\lambda$, the social surplus is strictly increasing with $\lambda$. Furthermore, $s^*$ is greater than zero, so trade inefficiency occurs with positive probability. Finally, from (9) it is immediately seen that the seller’s expected revenue is increasing with $\lambda$.

The characterization of the bidders’ payments is as follows. Recall that $U_i(0,0) = \int_{T_{-i}} \left\{ \sum_{k \in I} \alpha_{ik} [v_k x_k(v_{-i}) - x_k(v_i, v_{-i})] - \alpha_{ii} \int_0^{v_i} p_i(s_i, v_{-i}) \, ds_i \right\} \times f(v_{-i}) \, dv_{-i}, v_i \in T, i \in I$. To have $U_i(0,0) = 0$, we set the expression in braces at zero for all $(v_i, v_{-i})$. Therefore, fix state $v = (v_i, v_{-i})$, and suppose a winning bidder exists. Let $w \in I$ denote the winner. Then in equilibrium, for each state $v$ in which a winner exists, $x_j(v), j = 1, 2, \ldots, N$, solve the system of $N$ linear equations: $\sum_k \alpha_{jk} x_k(v_{-i}) = \alpha_{w} z(v_{-i})$, $\sum_k \alpha_{jk} x_k(v_{w}) = \alpha_{jw} v_{w}$, $j \in I \setminus \{w\}$. Finally, in those states in which the object is not assigned, all payments are zero.

The intuition behind the above results is as follows. Thanks to the individual participation constraint — non-negative total payoff — the seller can ask each losing bidder to pay an amount exactly equal to its ex post external surplus. Therefore, for encouraging truth-telling the seller lowers the winner’s payment with respect to the case of no ownership links — note that the weight of a unit payment in the winner’s payoff is 1 if it pays but only $\lambda$ if a losing bidder pays. In addition, the seller has less need to inefficiently renounce selling the object in order to encourage truth-telling — this explains why social surplus is increasing with $\lambda$.

**Example 4.** Let $N = 2, F(v) = v, v^+ = 1$. The optimal payment is $x_w(v_{w}, v_{-w}) = \max(\frac{\lambda v_{w}}{1 - \lambda^2}, x_{v_{-w}}(v_{w}, v_{-w}))$ and $x_{v_{-w}}(v_{w}, v_{-w}) = \lambda(\frac{v_{w}}{1 - \lambda^2} - \lambda^2 v_{w})$, where $w$ denotes the winning bidder.\(^8\) Moreover, from $c(s^*) = s^* - \frac{1 - s^2}{1 + \lambda^2} = 0$ we get $s^* = \frac{\lambda}{1 + \lambda^2}$; the optimal reserve price in the first-price auction is $b^*_w = \frac{\lambda}{1 + \lambda^2}$, and therefore, for $\lambda > 0$, it is $b^*_w > s^*$, which means that inefficiency is lower in the optimal selling procedure.

\(^8\) It is $x_w(v_{w}, v_{-w}) = \max(\frac{\lambda v_{w}}{1 - \lambda^2}, v_{w})$, and $x_{v_{-w}}(v_{w}, v_{-w}) = \lambda(\frac{v_{w}}{1 - \lambda^2} - \lambda^2 v_{w})$. Substituting the expression of $x_w(v_{w}, v_{-w})$ in that of $x_{v_{-w}}(v_{w}, v_{-w})$ yields the payments in the body text.
3 Extensions. Strategic entry in two-stage auction games

Until now we have assumed that all the firms take part in the auction. This is an innocuous assumption in a static setting, each firm will participate, possibly bidding zero, because entry does not entail any (opportunity) cost. However, let there be two stages in the game, an entry stage and a bidding one; and suppose it is common knowledge that in the second stage, at the moment of bidding, the number of active bidders is known to all players. Then, when there are ownership links, entry may reveal individual valuation, and hence a firm might find it profitable not to enter and let its payoff be determined by its shares in other firms. Let’s reconsider the first-price auction in this new setting.

3.1 The first-price sealed-bid auction

We assume that entries are simultaneous and that the number of active bidders is known to all players at the moment of bidding. In addition, we let $a_{ik} = a_{ih}$ for all $h$, and all $k, i = 1, \ldots, N$, so symmetry is preserved, no matter which bidders stay away. (Recall that $\alpha_{ik}$ denotes the non-negative weight firm $i$ assigns to the expected surplus of firm $k$.) Finally we assume that the seller’s reserve price can be made contingent on the number of active bidders. Firstly, we show that the seller’s revenue is not less than when entry is not strategic (the case we dealt with in Section 2.1). Let $b_m, m = 1, 2, \ldots, N$, denote the reserve price when $m$ bidders are active, and $b_N^*$ denote the optimal reserve price in the one-stage game. Now suppose $b_m = v^+$ for $m < N$, and $b_N = b_N^*$; then all the firms will participate, for no one will be better off staying away. This proves our assertion.

Secondly, we show that the seller’s revenue is decreasing with the concern parameter $\lambda$, just like under non-strategic entry. Two cases are to be considered. The first case, in which bidders learn their own valuations after entry, is easy. The above strategy — selling only if $N$ bidders enter — is optimal for the seller, since a bidder of a given valuation bids more aggressively when there are more bidders. Therefore, it is confirmed that the seller’s revenue in equilibrium is decreasing with the concern parameter $\lambda$. The second case, in which bidders learn their own valuations before entry, is more complex, because of the possibility of bidder self-selection according to valuation. For simplicity, we examine the case of two bidders only. When deciding whether to participate or not, bidder $i$ supposes that its rival adopts the monotone increasing (not strictly) participation function $\eta = \eta(v) : [0, v^+] \rightarrow [0, 1]$ — later we shall verify that this actually happens in equilibrium. Let $V_m(v_i)$ denote bidder $i$’s surplus when it wins, it is of type $v_i$ and the number of active bidders is $m = 1, 2$. Then bidder $i$ solves the following program:

$$
\begin{align*}
\text{Max } & \eta_i V_2(v_i) \int_0^{v_i} \eta(v) f(v) dv + \eta_i \lambda \int_{v_i}^{v^+} \eta(v) V_2(v) f(v) dv + \\
& \eta_i V_1(v_i) \int_0^{v^+} (1 - \eta(v)) f(v) dv + (1 - \eta_i) \lambda \int_0^{v^+} \eta(v) V_1(v) f(v) dv,
\end{align*}
$$

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where the maximand is proportional to its total payoff and \( \eta_i \) is its probability of entry.

We posit that each bidder’s payoff is increasing with its valuation — later we shall verify that this really occurs in equilibrium. Since bidders are symmetric, in equilibrium all of them use the same participation function. Now notice that bidder \( i \)’s optimal entry strategy is deterministic since the objective function is linear in \( \eta_i \) an increasing in \( v_i \); hence in equilibrium \( \eta(v_i) = 0 \) for \( v_i \) less than a cut-off value, which we call \( c \), and \( \eta(v_i) = 1 \) otherwise. Therefore, for \( v_i \geq c \)

\[
V_2(v_i) [F(v_i) - F(c)] + V_1(v_i) F(c) + \lambda \int_{v_i}^{v^+} V_2(v)f(v)dv - \\
\lambda \int_{v_i}^{v^+} V_1(v)f(v)dv \geq 0,
\]

(10)
The first three addends in the sum in (10) add up to type \( v_i \)’s (normalized) payoff in the case it enters, the last term is its (normalized) payoff in case it stays away. The bidding function when only one bidder is active equals the seller’s reserve price, \( b_1 \), for \( v_i \geq b_1 \), and equals zero otherwise; so \( V_1(v_i) = \max(0,v_i-b_1) \). Furthermore, letting \( B \) denote the equilibrium bidding function when two bidders are active, we have \( V_2(v_i) = v_i - B(v_i) \). (Of course, bidder \( i \) wins if \( B(v_i) > \max(b_2,B(v_{3-n})) \).) Proceeding as in Section 2 yields the bidding function \( B(v_i) = v_i - \frac{1}{\int_{v_i}^{v^+} F(v)f(v)dv} \int_{v_i}^{v^+} (F(v) - F(c))^1-\lambda dv, b_2 \leq v_i \leq v^+ \).

Notice that bidder \( i \)’s payoff is increasing with its valuation and strictly so for \( v_i \geq c \), as was assumed. Notice also that \( B(v_i) \) is increasing with \( c \), which means that, for given \( b_2 \), bidder self-selection induces more aggressive bidding — as was expected.

The seller chooses the values of the reserve prices and \( c \) that maximize \( U_s = 2 \int_{b_2}^{v^+} B(v) [F(v) - F(c)] f(v)dv + 2 [1-F(c)] F(c)b_1 \), under constraint (10).

In the appendix we prove that \( U_s \) is strictly decreasing with the concern parameter \( \lambda \). Therefore we can state the following.

**Proposition 4.** Consider a first-price sealed-bid auction, in which entry decisions are simultaneously and strategically made by cross-owned firms in a non-cooperative manner. Then:

a) The seller’s revenue is strictly decreasing with the concern parameter \( \lambda \).

b) When the reserve price is contingent on the number of active bidders, the seller’s revenue in equilibrium is not less than when entry is non-strategic.

It is noteworthy that, if the reserve price is not contingent on the number of active bidders, the seller’s revenue may fall with respect to the case of non-strategic entry. To see this, reconsider example 1 in Section 2.1. The bidding function is \( B(v) = \frac{(1-\lambda)v + c}{2} \), \( c \leq v \leq v^+ \), where \( c > b > 0 \), and \( b \) denotes the
seller’s reserve price.\textsuperscript{11} The equilibrium value of \( c \) is determined by the equation
\[
(c - b)c - \lambda \int_{c}^{v_{e}} (B(v) - b) \, dv = 0 \tag{11}
\]
Solving (11) yields the equilibrium value of \( c, c^{*} = c(b) \), which satisfies requirements for a cut-off value.\textsuperscript{12} The seller chooses the reserve price in order to maximize
\[
\int_{v_{e}}^{v_{s}} B(v)(v - c^{*}) \, dv + 2(1 - c^{*})c^{*}a.
\]
Figure 1 depicts the equilibrium for \( \lambda = 0.05 \). The seller’s revenue is lower than in Section 2.1 — the values are 0.412.55, and 0.414.92, respectively. Also social surplus is lower, since the equilibrium value of \( b \) is greater than \( b^{*} \) — the respective values are 0.506.24 and 0.504.77. Numeric simulations suggest that these qualitative results hold true for all values of \( \lambda \).

\[<< \text{Insert Figure 2 here >>} \]

### 3.2 On the optimal selling procedure

We only observe that all the firms will certainly participate if the seller announces that the object will not be sold if a firm stays away. Therefore, the following holds true.

\textit{Proposition 5.} Consider the optimal selling procedure for the case in which entry decisions are simultaneously and strategically made by cross-owned firms in a non-cooperative manner. Then, the seller’s revenue is not less than when entry is non-strategic.

### 4 Summary and conclusions

In this paper we have examined the effect of passive ownership links among bidders in auctions. The analysis has focused on the case of symmetric bidders with no collusion. Initially, it has been assumed that all potential bidders participate in the auction, provided a bidder’s total expected payoff is non-negative. Firstly, we proved that: a) ownership links weaken the willingness to bid high and thus hurt the seller in first-price, in second-price, and in all-pay auctions; b) such auctions are not revenue equivalent; c) social surplus strictly decreases as an effect of the seller’s reaction, and also the bidders’ own surplus may decrease.

Next, we characterized the revenue-maximizing selling procedure: in sharp contrast with standard auctions, the result is that both the seller’s revenue and social surplus are strictly increasing with the intensity of ownership links. The reason for this surprising result is that standard auctions impose a more restrictive individual participation constraint than the optimal selling procedure: in the former a bidder’s own surplus must be non-negative, in the latter only the total payoff must be non-negative. Therefore, in the optimal auction, the seller is able to extract surplus from the losing bidders and has less need to destroy potential surplus for encouraging truth-telling.

\textsuperscript{11} \( c = 0 \) is unfeasible, since this requires \( \int_{b}^{v_{e}} (|v - B(v)|) \, dv - \int_{b}^{v_{e}} (v - b) \, dv \geq 0 \), which is impossible.

\textsuperscript{12} It is \( c^{*} = \frac{1}{\lambda(\frac{1}{1 - 2\lambda} + 2)} \left( b + \frac{\lambda}{2 - \lambda} + b\lambda + \sqrt{\frac{\lambda^{2}(1-b)^{2} - \lambda(b^{2} - 2b + 2) - 2b}{2 - \lambda}} \right) \).
In the second part of the paper, we dispense with the assumption that all potential bidders participate, and explore the consequences of bidder self selection in a two-stage game, in which the potential bidders first decide to enter the auction or stay away, and then the auction is held, all players knowing the number of active bidders. With reference to the first-price sealed-bid auction, it is confirmed that the seller’s revenue is strictly decreasing with the intensity of ownership links. As for the optimal selling procedure, it results that the seller’s revenue is strictly increasing with the intensity of ownership links, provided he can commit to the strategy of selling if and only if all potential bidders participate.

A possible extension of the paper would consider asymmetric bidders, characterized by ownership links of different intensity. Although a formal analysis is left for future research, intuitively it is clear that ownership links decrease bids in this case as well: the bidders with strong links bid less aggressively for the reasons explained in the paper, those with weak links do the same because they have less need to bid high in order to beat their opponents. As regards the optimal selling procedure, it is clear that the seller can extract surplus selectively from each bidder, so the seller certainly gains under asymmetric ownership links as well.
References


Appendix

Proof of Proposition 1 (Section 2.1.)

a) It is a trivial consequence of the fact that $B$ is strictly increasing with $v$.

b) $b^*_s$ maximizes the seller’s expected revenue $U_s = N \int_{b_s}^{v_s} B(v) F^{N-1}(v) f(v) dv$.

Thus it solves the following equation:

$$\int_{b_s}^{v_s} B_s(v) F^{N-1}(v) f(v) dv - b_s F^{N-1}(b_s) f(b_s) = 0,$$

if the term in square brackets is negative. If it results that, for any given $\lambda$, $\lim_{b_s \to 0} \frac{\partial U_s}{\partial b_s}$ is strictly positive, then $b_s = 0$ can never be optimal. It is easily shown that $B_s(v)$ is always positive and strictly positive for $v > b_s$, thus the desired result is proven.

c) As to the seller’s revenue, it is in equilibrium $\frac{dU_s}{dv} \bigg|_{v_s = b^*_s} = \frac{dU_s}{dv} \bigg|_{v_s = b^*_s} = N \int_{v_s}^{b^*_s} \frac{B_s(v)}{F(v)} F^{N-1}(v) f(v) dv$. The desired result is proven if we show that, for any given $\lambda$, $\frac{\partial B_s}{\partial v}$ is negative for all $v$, and strictly negative for $v > b^*_s$. We can use the following lemma (Gronwall’s): Let $I \subset \mathbb{R}$ be an interval and $x_0 \in I$. Let $u, v : I \rightarrow \mathbb{R}$ be two continuous functions on $I$, $u$ non-negative and $c \geq 0$. If $v(x) \leq c + \int_{x_0}^{x} u(t) v(t) dt$ for all $x \in I$, then $v(x) \leq c \exp(\int_{x_0}^{x} u(t) dt )$.

(For a proof see Hartman (1964), Ch. 3). Now, for any $v \geq b^*_s$, we get from (4):

$$B_s(v) = \int_{v}^{b^*_s} (z-B(z))^\lambda f(z) dz + \lambda \int_{v}^{b^*_s} B_s(z) (z-B(z))^\lambda f(z) dz$$

and therefore $B_s(v) \leq \int_{v}^{b^*_s} \frac{\lambda f(z)}{f(z)} B_s(z) dz$. Since $B_s$ and $\frac{\lambda f(z)}{f(z)}$ are continuous, we can apply Gronwall’s lemma and conclude from the non-negativity of $f$ that $B_s(v)$ is strictly negative for $v > b^*_s$ and is zero for $v = b^*_s$.

As to the joint surplus in equilibrium, its derivative with respect to $\lambda$ is opposite in sign to $\frac{dU_s}{dv} = - \left( \frac{\partial^2 U_s}{\partial v^2} \frac{\partial^2 U_s}{\partial b_s} \right) \bigg|_{b_s = b^*_s}$ . As $U_s$ is locally strictly concave, by assumption, it is sign $\left( \frac{dU_s}{dv} \right) = \text{sign} \left( \frac{\partial^2 U_s}{\partial v^2} \frac{\partial^2 U_s}{\partial b_s} \right) \bigg|_{b_s = b^*_s} = \text{sign} \left( \int_{v}^{b^*_s} B_s(v) (F(v))^{N-1} f(v) dv \right) \bigg|_{b_s = b^*_s}$. For any $v \geq b_s$, we get from (5)

$$B_{b,\lambda}(v) = \frac{\partial F(b_s)}{\partial \lambda} (F(v))^{N-\lambda} > 0.$$ 

Therefore we conclude that $\frac{dU_s}{dv} > 0$.

Remark to Proposition 1

Following is an example in which, contrary to example 1 in Section 2.1., $S^B$ is always increasing with $\lambda$. There are two bidders, and $v$ is distributed in the interval $[0, 1]$ according to the function $F(v) = v^8$. It results $b^*_s = (9 + 8\lambda)^{-\frac{1}{8}}$, $B^*(v) = \frac{8c(1-\lambda)+v^{-8(1-\lambda)(9+8\lambda)\frac{1}{\lambda}}{(9+8\lambda)}$, and finally $S^B = \frac{17(9+8\lambda)^{\frac{8\lambda}{17(9+8\lambda)+8(1-\lambda)}-8(1-\lambda)}}{17(9+8\lambda)^{\frac{8\lambda}{17(9+8\lambda)+8(1-\lambda)}}}$, which is strictly increasing with $\lambda$.

Proof of Proposition 2 (Section 2.2.)

a) We have to prove that proposition 1 holds true for the second-price sealed-bid auction.

Firstly, we show that $B_\lambda(v_1, b_s) < 0$ for $b_s \leq v_i < v^+$. This is easily seen via Gronwall’s lemma. Indeed, we can write $B(v_i) = b_s + \int_{b_s}^{v_i} (v - B(v)) \frac{f(v)}{F(v)} \frac{(1-\lambda)(N-1)}{\lambda} dv$, and hence $B_\lambda(v_1) = - \int_{b_s}^{v_i} B_s(v) \frac{f(v)}{1 - (F(v))} \frac{(1-\lambda)(N-1)}{\lambda} dv - \int_{b_s}^{v_i} (v - B(v)) \frac{f(v)}{1 - (F(v))} \frac{(N-1)}{\lambda} dv$. 

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Now, for any \( v_i \in (b_s, v^+) \) \( B_\lambda(v_i) < \left| \int_{b_s}^{v_i} f(v) \frac{(N-1) \lambda}{\lambda - 1} dv \right| \), which implies that \( B_\lambda(v_i) \) is strictly negative. Since \( \frac{dU_s}{d\lambda} \big|_{b_s = b_s^*} = \frac{dU_s}{d\lambda} \big|_{b_s = b_s^*} \) is equal in sign to \( B_\lambda(v_i) \), we conclude that the seller’s revenue is decreasing with \( \lambda \).

Next, notice that, if an optimal solution with \( v_i = b_s \), \( B_{\lambda_b}(v_i) = (N-1) \int_{b_s}^{v_i} f(v) - \left( \frac{(N-1) \lambda}{\lambda - 1} \right) \int_{b_s}^{v_i} B_{\lambda_b}(v) \frac{1}{1-F(v)} dv. \)

Applying again Gronwall lemma yields \( B_{\lambda_b}(v_i) > 0 \). Since in equilibrium \( \frac{dU_s}{d\lambda} \) is equal in sign to \( B_{\lambda_b}(v_i) \), we conclude that \( \frac{dU_s}{d\lambda} > 0 \).

The seller’s expected revenue in equilibrium is strictly decreasing with \( \lambda \) (Section 3.1.)

Notice first that \( c = 0, b_1 = v^+, b_2 = b_s^* \) is a feasible solution to the seller’s problem. In fact, when \( c = 0 \), (10) becomes, for all \( v_i \), \( v_i - B(v_i) ) F(v_i) I_{[b_2, v^+]}(v_i) + \lambda \int_{b_2}^{v^+} [v - B(v)] f(v) dv - \lambda \int_{b_1}^{v^+} (v - b_1) f(v) dv \geq 0 \), where \( I_{[b_2, v^+]}(b_1) \) equals 1 for \( b_1 \in [b, v^+] \), and equals zero otherwise. This further implies \( \int_{b_2}^{v^+} [v - B(v)] f(v) dv \geq \int_{b_1}^{v^+} (v - b_1) f(v) dv \), which is satisfied for \( b_1 = v^+, b_2 = b_s^* \).

Next, notice that, if an optimal solution with \( c > 0 \) exists, then it must be \( b_1 < c < b_2 \). To see this, suppose \( b_2 \) is given, and take \( b_1 \) to be such that (10) is satisfied as an equality with \( v_i = c \) and \( c = 0 \) — this requires \( b_1 > b_2 \). Then let \( b_1 \) decrease: (10) can be satisfied for \( v_i = c \) only if \( c > b_1 \), which implies \( b_1 < b_2 \).

Therefore, it is

\[
(c - b_1) F(c) - \lambda \int_{b_2}^{v^+} (v - b_1) f(v) dv - \lambda \int_{b_2}^{v^+} (B(v) - b_1) f(v) dv \geq 0
\]

with equality if \( c > b_1 \). (1a)

Thanks to this result, we can replace (10) with (1a) in the seller’s problem. Finally, notice that the partial derivative, with respect to \( \lambda \), of the left side of (1a) is negative, while that of \( U_s \) is strictly negative; therefore, by the envelope theorem, \( U_s \) in equilibrium is strictly decreasing with \( \lambda \). This proof generalizes to the case of \( N \) bidders — we simply repeat the main argument.
Figure 1. The seller’s expected revenue as a function of $\lambda$. $N = 2, F(v) = v, v \in [0, 1]$. The dashed curve depicts the first-price sealed-bid auction, the solid curve depicts the second-price sealed bid auction, and the dot curve depicts the all pay auction.
Figure 2. The equilibrium in the first-price sealed-bid auction with strategic entry and one reserve price. $N = 2, F(v) = v, v \in [0, 1], \lambda = \frac{5}{100}$. The U-shaped curve is an isoquant of the objective function, the dotted curve is the participation constraint (see eq. 10).