Multi-moment Approximate Option Pricing Models: A General Comparison (Part 1)*

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Abstract

After the seminal paper of Jarrow and Rudd (1982), several authors have proposed to use different statistical series expansion to price options when the risk-neutral density is asymmetric and leptokurtic. Amongst them, one can distinguish the Gram-Charlier Type A series expansion (Corrado and Su, 1996-b and 1997-b), log-normal Gram-Charlier series expansion (Jarrow and Rudd, 1982) and Edgeworth series expansion (Rubinstein, 1998). The purpose of this paper is to compare these different multi-moment approximate option pricing models. We first recall the link between risk-neutral densities and moments in a general statistical series expansion framework. We then derive analytical formulae for these different four-moment approximate option pricing models, namely, the Jarrow and Rudd (1982), Corrado and Su (1996-b and 1997-b) and Rubinstein (1998) models. We investigate in particular the conditions that ensure the respect of the martingale restriction (see Longsta¨ff, 1995) and compare with option pricing models such as Black and Scholes (1973) and Hermite polynomial models (see Madan and Milne, 1994, Abken et al., 1996). We also get for these approximate option pricing models analytical expressions of implied probability distribution, implied volatility smile functions and several hedging parameters of interests, such as the Psi and the Khi that measure respectively the changes in the option price with respect to the changes in kurtosis and skewness.

Keywords: Option Pricing Models, Stochastic Volatility, Skewness, Kurtosis.


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1 Introduction

The Black and Scholes (1973) formula is certainly one of the most used in finance but presents some inconsistencies. In particular, several empirical studies\(^1\) show that the model missprices deep out-of-the-money and deep in-the-money options. In other words, when the Black-Scholes formula is inverted, the implied volatilities estimates differ across exercise prices and maturities, and form patterns called “smile”, “smirk” or “sneer” depending on their shapes.\(^2\) This result is generally attributed to the unrealistic hypothesis of a geometric Brownian motion for the underlying asset process or, equivalently, of a normally distributed continuous rate of return with constant volatility under an equivalent martingale measure. Indeed, if rare events are more frequent than it is assumed in the normal case, then the price of deep out-of-the-money options will be higher than the Black and Scholes (1973) model predicts. If, moreover, the log-return distribution is negatively skewed, prices of deep out-the-money put options will be higher than those of deep in-the-money call options and the implied volatility function will be downward biased.

In order to avoid these biases, different approaches have been proposed. A first one is to consider alternative stochastic processes than the geometric Brownian motion with or without additional stochastic factors. For instance, a jump-diffusion process are chosen by Merton (1976) and more recently by Bates (1991 and 1996-a), whilst Hull and White (1987), Stein and Stein (1991) and Heston (1993) consider stochastic volatility models. Bates (1996-b and 2000) and Pan (2002) extend the jump-diffusion model to incorporate stochastic volatility to explain the structure of option prices while Bakshi \textit{et al.} (1997 and 2000) develop option pricing models that admit simultaneously stochastic volatility, stochastic interest rate and random jump.

A second approach is to use binomial - or trinomial - lattices in order to approximate the whole structure of market prices (see Rubinstein, 1994, Derman and Kani, 1994, Dupire 1994 and Dernan \textit{et al.}, 1996, Jackwerth, 1997). In achieving an exact cross-sectional fit of option prices, trees can be constrained to reproduce moments of a prespecified implied density (see Rubinstein, 1998, Li, 2000, Ang \textit{et al.}, 2001).

Despite the fact that both approaches can yield skewed and leptokurtic risk-neutral density, they are not perfectly satisfactory. The most severe critics of these related models are the lack of parcimony (leading to possible overfitting), the choice of a deterministic volatility function (see Dumas \textit{et al.}, 1998), the

\(^1\)See, for instance, MacBeth and Merville (1979), Rubinstein (1985 and 1994) and Bates (1996-c).

\(^2\)Bates (2000) shows that “smiles” often appear before the crash of 1987 on the American market, whilst “sneer” patterns are more likely to be found since.
existence of inadequate volatility term structure (see Das and Sundaram, 1999). Moreover estimation problems on illiquid markets are reported.


Thirdly, semi-parametric models consist in approximating the state price density using empirical counterparts of the implied moments. Initially developed by Jarrow and Rudd (1982), this last approach aims to approximate the risk-neutral density by a statistical series expansion such as a Gram-Charlier Type A series expansion (Corrado and Su, 1996-b and 1997-b, Backus et al., 1997, Bouchaud et al., 1998, Brown and Robinson, 1999, Knight and Satchell, 2001), a log-normal Gram-Charlier series expansion (Jarrow and Rudd, 1982, Turnbull and Wakeman, 1991, Corrado and Su, 1996-a, 1997-a, Jondeau and Rockinger, 2000, Flamouris and Giamouridis, 2002) or an Edgeworth series expansion (Rubinstein, 1998, Li, 2000).\(^4\) The series are truncated to a finite order that usually gives a tractable closed-form expressions for option prices. In this last approach, the risk-neutral skewness and kurtosis of the underlying asset enter in option pricing in a very natural way since the coefficients of statistical series expansions are functions of moments of the given and approximating distributions.

The purpose of this article is to focus on this last field of literature. We aim to present, in an unified framework, the theoretical foundations of the option pricing models based on statistical series expansion methods, namely, the Jarrow and Rudd (1982), the Corrado and Su (1996-b and 1997-b) and the Rubinstein (1998) models.

Our study provides several contributions. First, we investigate the conditions that ensure the respect of the martingale restriction (see Longstaff, 1995). This gives us crucial insights on approximations involved in the multi-moment

\(^3\) Indeed for a given expiration date, there exists an infinite number of stochastic processes which are consistent with one particular risk-neutral distribution (see, for instance, Melick and Thomas, 1997 and Dapire, 1998).

\(^4\) While these expansions are the most popular in the literature, others have also been considered such as Laguerre series expansions (Brenner and Eom, 1997 and Dufresne, 2000) and Kummer functions (Abadir and Rockinger, 1997).
approximate option pricing models. Indeed, while it is showed that the mar-
ingale restriction is always fulfilled in the Jarrow and Rudd (1982) case, this
might not be the case for the Corrado and Su (1996-b and 1997-b) and the
Rubinstein (1998) models. We also establish the link between these models and
alternative option pricing models such as Black and Scholes (1973) and Hermite
polynomial models (see Madan and Milne, 1994, Abken et al., 1996). Next,
we provide analytical formulae for implied density function and we generalize
the approach of Backus et al. (1997) regarding the volatility smile function.
We finally provide hedging parameters of interests following Corrado and Su

The paper is organized as follows. In section 2, we review the statistical
foundations and the pricing formulae of the Jarrow and Rudd (1982), Corrado
and Su (1996-b and 1997-b) and Rubinstein (1998) models. In section 3, we
present the implied probability density and the implied volatility smile functions.
We also computes the Greeks - namely, the Delta, Gamma, Vega, Khi and Psi5.
Section 4 summarizes and concludes. Main proofs and figures are collected in
Appendixes.

2 Pricing of Options when Risk-neutral Densi-
ties are Skewed and Leptokurtic

The first element of interest in option pricing is the conditional distribution of
the terminal price of the underlying asset. Let \( x_{\tau} \) be the \( \tau \)-th period log-return
on the underlying asset defined such as:

\[
x_{\tau} = \ln \left( \frac{S_T}{S_t} \right) = \ln \left( \frac{S_{t+i\Delta}}{S_{t+(i-1)\Delta}} \right) = \sum_{i=1}^{N} x_i
\]

where \( S_T \) and \( S_t \) are respectively the terminal and the actual price of the under-
lying asset, \( N = \tau / \Delta \) is the number of unit time intervals of length \( \Delta \) during
a period \( \tau = (T - t) \), and \( x_i \) is the instantaneous log-return on the underlying
asset. Rearranging terms, we obtain:

\[
\ln S_T = \ln S_t + \sum_{i=1}^{N} x_i
\]

The two last one - proposed by Hull and White (1997) - measure respectively changes in
the option price with respect to changes in skewness and kurtosis.
then:

\[ S_T = S_t \exp \left( \sum_{i=1}^{N} x_i \right) \]  

(3)

and the conditional distribution of the terminal price of the underlying asset depends on that of \( x_i \). If we assume that \( x_i \) are IID random variables with finite variance, it follows by application of the central limit theorem and the definition of a log-normal random variable\(^6\) that when \( N \) tends to infinity, the underlying asset terminal log-price is conditionally normally distributed and the underlying asset terminal price is conditionally log-normally distributed.

The second element of interest when valuing options is the determination of the fair price in a risk-neutral framework. An European call option is a contract which confers on its holder the right, with no obligation, to purchase an underlying asset, which current price is noted \( S_t \), for a prescribed amount, known as the exercise or strike price, denoted \( K \), at the expiration date, \( T \). Under the assumptions of complete market and no arbitrage opportunity, and if we suppose that the risk-free rate of interest, denoted \( r \), is constant, the theoretical price of a call option is the present value of the expected payoff at expiry, given by the following pricing kernel (see Cox and Ross, 1976):

\[
C = C [S_t, K, \tau, r, f, S_T, \theta] = e^{-r\tau} E_Q [\text{Max} (S_T - K, 0)]
\]

(4)

\[
e^{-r\tau} \int_{S_T=K}^{+\infty} (S_T - K) f (S_T) dS_T
\]

where \( E_Q [\cdot] \) is the expectation under the risk-neutral probability measure, \( \theta \) is a vector of parameters - the first moments - characterizing the risk-neutral density of underlying asset terminal price \( f (S_T) \). A closed-form for the option formula can then be obtained if we assumed a log-normal distribution for the terminal price of the asset, as in Black and Scholes (1976), or if we use a statistical series expansion for the conditional density of the price of the asset, as in Jarrow and Rudd (1982) or of the related continuously compounded return, as in Corrado and Su (1996-b and 1997-b) and Rubinstein (1998).

2.1 Risk-neutral Density and Moments

The problem is then to get an analytical expression for the risk-neutral density function. One way of doing that is, following Jarrow and Rudd (1982), to use a statistical series expansion\(^7\) of the state price density in order to get an approximation used in (4) when replacing \( f (x) \) by:

\[
f (x) = v (f, g, x, \theta) + \varepsilon (x)
\]

(5)

\(^6\)A random variable \( x \) is said to be log-normal if \( \ln(x) \) is normally distributed. For a study of the log-normal distribution, see, for instance, Aitchinson and Brown (1966).

\(^7\)Statistical series expansion are conceptually similar to a Taylor series expansion: a given density is approximated by an expansion around a prespecified distribution.
where $g(\cdot)$ is a fitted density, $x$ the random variable under interest - terminal price or log-return - $\theta$ is a vector of moments characterizing the “true” risk-neutral density, $v(\cdot)$ a statistical series expansion and $\varepsilon(x)$ a residual.

In this case, estimations of parameters included in the vector of moments $\theta$ are sufficient to have a parametric approximation of the risk-neutral density. More formally, any robust class of density $f(x)$ can be written as (see Johnson et al., 1994, p.28 and Appendix 1):

$$f(x) = g(x) + \sum_{i=0}^{\infty} \left[ \sum_{j=1}^{N-1} (-1)^j \frac{k_j D^j}{j!} \right] g(x) + \varepsilon(x) \quad (6)$$

where $g(x)$ is an arbitrary density and $\kappa_j(\cdot), j = [1, ..., N]$ its cumulants, and with $k_j = [\kappa_j(f) - \kappa_j(g)]$, $\kappa_1(\cdot) = \mu_1(\cdot)$; $\kappa_2(\cdot) = \mu_2(\cdot)$; $\kappa_3(\cdot) = \mu_3(\cdot)$; $\kappa_4(\cdot) = \mu_4(\cdot) - 3\mu_2(\cdot)^2$ where $\mu_j, j = [1, 2, 3, 4]$ are the centered moments of order $j$, $D$ is the differentiation operator such as $D^j g(x) = d^j g(x)/dx^j$ and $\varepsilon(x)$ is a residual.

In the last formula, the terms in $g(x)$ represents a traditional general statistical series expansion. Some restrictions could be added on existence of moments and on the fact that the distribution could be uniquely defined using its moments. Specific ordering of terms and special choices of the form of the approximating distribution lead to several writings of equation (6).

The way ordering terms in the general form lead to different statistical series expansion as presented hereafter. Indeed, developing and collecting terms determined by successive derivatives of $g(x)$ in (6), up say to the fourth order, might lead to several representations of the approximating distribution. Some of these include Cornish-Fisher series expansion and Johnson family of curves. The cumulants of $f(x)$ are defined as coefficients of $(j)^{-1} d^j g(x)/dx^j$ in equation (6), whether or not $f(x) \geq 0$. So expression (6) will not in general constitutes a proper probability density function (see Kendall and Stuart, 1977, pp. 168-171, Johnson et al., 1994, pp. 25-30). Nevertheless, this problem can be solve by imposing restrictions on the domain of variation of the moments (see for instance, Barton and Dennis, 1952, Balitskaia and Zolotuhina, 1988, and Jondeau and Rockinger, 2001). Another problem that can arise is that, even if for all $x, f(x) \geq 0$, it may display multimodality (see Barton and Dennis, 1952). Despite these limitations, it is often possible to obtain from statistical series expansion useful approximate expression of a distribution with known moments.

8 Some of the others common approximation techniques of density by their moments include Cornish-Fisher series expansion and Johnson family of curves.

9 The cumulants of $f(x)$ are defined as coefficients of $(j)^{-1} d^j g(x)/dx^j$ in equation (6), whether or not $f(x) \geq 0$. So expression (6) will not in general constitutes a proper probability density function (see Kendall and Stuart, 1977, pp. 168-171, Johnson et al., 1994, pp. 25-30). Nevertheless, this problem can be solve by imposing restrictions on the domain of variation of the moments (see for instance, Barton and Dennis, 1952, Balitskaia and Zolotuhina, 1988, and Jondeau and Rockinger, 2001). Another problem that can arise is that, even if for all $x, f(x) \geq 0$, it may display multimodality (see Barton and Dennis, 1952). Despite these limitations, it is often possible to obtain from statistical series expansion useful approximate expression of a distribution with known moments.

10 In a financial framework, expansions consider usually only the first four moments.

11 That is not the case for the log-normal distribution for instance.
we obtain:

\[
 f(x) = v_{GC}(f, g, x, \theta) + \varsigma(x)
\]

\[
 = g(x) - k_1 \frac{dg(x)}{dx} + \left[ \frac{k_2 (k_1)^2}{2!} \right] \frac{d^2 g(x)}{dx^2}
\]

\[
 - \left[ \frac{k_3 + 3k_1 k_2 + 3 (k_1)^3}{3!} \right] \frac{d^3 g(x)}{dx^3}
\]

\[
 + \left[ \frac{k_4 + 4k_3 k_1 + 3 (k_2)^2 + 6 (k_1)^2 k_2 + (k_1)^4}{4!} \right] \frac{d^4 g(x)}{dx^4}
\]

\[
 + \varsigma(x)
\]

where \( k_j \), with \( j = [1, ..., 4] \), are defined as previously and \( \varsigma(x) \) is an error term.

The state price density is then a linear combination of \( g(x) \) and its derivatives. The collection of terms in \( g(x) \) is called a Gram-Charlier series expansion (see for instance Johnson et al., 1994, p.28)\(^{12}\). Second, third, fourth and fifth terms in equation (7) allow to adjust \( g(x) \) according to the gap between respectively, the mean, the variance, the skewness and the kurtosis of the distribution and that of the approximating density function (each term being weighted by the first, second, third and fourth derivatives of the approximating density function). The last part of equation (7) - the residual \( \varsigma(x) \) - captures terms neglected in the expansion.

If we assume moreover that \( x \) is a standardized random variable and \( g(x) \) a Gaussian distribution, then equation (7) becomes:

\[
 f(x) = v_{GC}(f, \varphi, x, \theta) + \varsigma(x)
\]

\[
 = \varphi(x) + \frac{\kappa_3(f)}{3!} H_3(x) \varphi(x) + \frac{\kappa_4(f)}{4!} H_4(x) \varphi(x) + \varsigma(x)
\]

where \( \varphi(x) = (2\pi)^{-1/2} \exp(-x^2/2) \) is the standard normal density function, \( \kappa_j(\varphi) = \kappa_j(f) \) for \( j = [1, 2] \) and \( \kappa_j(\varphi) = 0 \) for \( j = [3, 4] \). \( H_i(x) \) denotes the \( i \)-th Hermite polynomial defined by Rodrigues’ formula\(^{13}\) \( H_i(x) = (-1)^i \varphi(x)^{-1} d^i \varphi(x)/dx^i \) and \( \varsigma(x) \) is a residual.\(^{14}\) The equation (8) corre-

\(^{12}\) Some authors refer to it also as a Bruns-Charlier Expansion (see Hall, 1997).

\(^{13}\) See Abramowitz and Stegun (1972).

\(^{14}\) The Hermite polynomials through the fourth order are (see Kendall and Stuart, 1977, p.163):

\[
\begin{align*}
 H_0(x) &= 1 \\
 H_1(x) &= x \\
 H_2(x) &= (x^2 - 1) \\
 H_3(x) &= (x^3 - 3x) \\
 H_4(x) &= (x^4 - 6x^2 + 3) \\
 H_5(x) &= (x^5 - 10x^3 + 15x) \\
 H_6(x) &= (x^6 - 15x^4 + 45x^2 - 15)
\end{align*}
\]
spends to Gram-Charlier Type A series or Hermite polynomial series expansion.15

For practical purposes, expression (6) is usually truncated up to the fourth order, and the remainder $\varepsilon(x)$ is dropped. Since the successive terms in a Gram-Charlier expansion are not necessarily in decreasing order of importance, $\varsigma(x)$ in (7) may not converge uniformly to zero as more terms are added. However, if $x$ is a normalized sum of $n$ independent and identically distributed random variables $x_i$, with $i = [1, \ldots, n]$, that is:

$$x = \sigma^{-1} \sum_{i=1}^{n} (x_i - \mu)$$

it is possible to arrange differently terms in equation (6) such as to ensure that it constitutes a proper asymptotic series expansion16. The ordering is based on the fact that for a sum of $n$ IID standardized random variables17, the $j$-th cumulant is proportional to $n^{1-j/2}$, with $j \geq 2$ (see Appendix 2). Developing and collecting terms of equal order in $n^{-1/2}$ in (6), say up to $n^{-1}$ order, $f(x)$ can then be expressed as:

$$f(x) = v_E(f, g, x, \theta) + \xi(x)$$

$$= g(x) - n^{-1/2} k_{i,3} \frac{d^3 g(x)}{dx^3}$$

$$+ n^{-1} \left[ \frac{k_{i,4}}{4!} \frac{d^4 g(x)}{dx^4} + 10 \left( \frac{k_{i,3}}{6} \right)^2 \frac{d^6 g(x)}{dx^6} \right]$$

$$+ \xi(x)$$

where $k_{i,j} = [\kappa_{i,j}(f) - \kappa_{i,j}(g)]$ with $\kappa_{i,j}$ is now the $j$-th cumulant of the standardized random variable $\sigma^{-1} (x_i - \mu)$, $k_{i,1} = 0$ and $k_{i,2} = 1$, with $(i \times j) = (1, \ldots, n) \times (3,4)$, and $\xi(x)$ is a residual with $\xi(x) = o \left( n^{-1} \right)$ where $o(.)$ is the landau notation.

In the last formulation, the group of terms in $g(x)$ is known as an Edgeworth series expansion (see, for instance, Johnson et al., 1994, p.28)18. Second and third terms in equation (10) allow to adjust $g(x)$ according to the gap between the skewness and the kurtosis of the risk-neutral distribution function and that of the approximating density (successive terms being now weighted by $n^{-1/2}$ and $n^{-1}$). The last part of equation (10) - the residual $\xi(x)$ - takes into account terms in the development based on higher order cumulants.

15 While formulae (8) is one of the most commonly used in statistical theory, it must be emphasized that Gram-Charlier expansion based on a standard beta, standard gamma, poisson, log-normal (see infra) and t-student distributions have also been developed.

16 An asymptotic expansion is defined to be a expansion which has the property that when truncated at some finite number $r$, the remainder is of smaller order than the last term that has been included (see for instance, Hall, 1992, p.45 and Spanos, 1986, pp.205-206).

17 When standardized random variables are not an IID sequence see Kochard (1999).

18 Some authors refer to it also as a Edgeworth-Sargan (Mauleon and Perote, 2000).
If we assume again that \( g(x) \) is a standard normal density, equation (10) becomes:

\[
\begin{align*}
    f(x) &= v_E(f, \varphi, x, \theta) + \eta(x) \\
    &= \varphi(x) + \frac{\kappa_3(f)}{3!} H_3(x) \varphi(x) \\
    &\quad + \left[ \frac{\kappa_4(f)}{4!} H_4(x) + 10 \frac{[\kappa_3(f)]^2}{6!} H_6(x) \right] \varphi(x) \\
    &\quad + \eta(x)
\end{align*}
\]

where \( \varphi(x) \), \( H_i(x) \) and \( \eta(x) \) denotes respectively the standard normal density, the \( i \)-th Hermite polynomial and a residual. This form is called a normal Edgeworth series expansion (see Spanos, 1986).

Note that none of the expression (10) or (11) possess a general theoretical superiority over the equation (7) or (8) since they depend on a particular assumption about the orders of magnitude of successive cumulants (see Johnson et al., 1994, p.28).

2.2 Fourth-moment Option Pricing Models

The statistical series expansion methodologies recalled, it is then possible to present the derivation of the approximate multi-moment option pricing models, depending on the choice of the approximating distribution in the statistical series expansion of the risk-neutral density.

2.2.1 The Black and Scholes (1973) Model

Black and Scholes (1973) model assumes that the dynamics of the underlying asset follows a geometric Brownian motion:\(^{19}\)

\[
dS_t = \left( \mu + \frac{\sigma^2}{2} \right) S_t dt + \sigma S_t dW_t
\]

(12)

with \( \mu \) is the expected value of \( \ln(S_T/S_t) \), \( \sigma \) represents the related volatility and \( W_t \) is a standard Brownian motion.

When markets are complete, Harrison and Pliska (1981) show that there exists a risk-neutral transformation that leads to the following expression:

\[
dS_t = \tau S_t dt + \sigma S_t dW^Q_t
\]

(13)

where \( W^Q_t \) is a Brownian motion under the risk-neutral probability measure.

It follows from Itô’s lemma that the risk-neutral density of the terminal price of the underlying asset is log-normal that is:

\[
f(S_T) = \frac{1}{S_T \sigma \sqrt{2\pi \tau}} \exp \left\{ - \frac{\left[ \ln \left( \frac{S_T}{S_0} \right) - (\tau - \frac{1}{2} \sigma^2) \tau \right]^2}{2\sigma^2 \tau} \right\}
\]

(14)

\(^{19}\)For the stochastic differential equation notation, see Baxter and Rennie (1996), p.85.
or, by definition, that of the log-normal asset return is normal, that is:

\[ f (\ln S_T) = \frac{1}{\sigma \sqrt{2\pi \tau}} \exp \left\{- \frac{[\ln S_T - (\ln S_t + [r - \frac{1}{2} \sigma^2] \tau)]^2}{2\sigma^2 \tau} \right\} \]  

(15)

so the price of an European call option under the Black and Scholes (1973) assumptions could be written as:

\[ C_{BS} = C [S_t, K, \tau, r, v, f, g, x, \theta] = e^{-r\tau} \int_{S_T=K}^{\infty} (S_T - K) v [f(S_t, \theta), g(S_t, \theta)] dS_T \]  

(16)

where \( l(.) \) is the log-normal distribution and \( v(f, g, \theta) \) is defined - in the particular case of Black and Scholes (1973) - such as:

\[
\begin{aligned}
  v(\cdot) &= Id \\
  f(\cdot) &= l(\cdot) \\
  x &= S_T \\
  g(\cdot) &= f(\cdot) \\
  \theta &= \sigma
\end{aligned}
\]

Performing the following change of variable on \( S_T \) in integral (16):

\[ z = \frac{\log \left( \frac{S_T}{S_t} \right) - \mu \tau}{\sigma \sqrt{\tau}} \]  

(17)

leads to the Black and Scholes formula (1973), that is:

\[ C_{BS} = e^{-r\tau} \int_{z=\ln(K/S_t)-\mu \tau}^{\infty} (S_T - K) \varphi(z) dz \]

(18)

with:

\[ d = \log \left( \frac{S_t}{Ke^{-r\tau}} \right) + \sigma^2 \tau / 2 \]

\[ \sigma \sqrt{\tau} \]

where \( \varphi(.) \) and \( \Phi(.) \) are respectively the standard normal density function and the standard normal distribution.

The main advantage of this model is that all parameters, except the volatility, are directly observable. However, empirical evidence against the hypothesis that returns are homoskedastic and normally distributed, and the existence of some anomalies on option markets reported in several studies (see for instance Rubinstein, 1994) lead to the development of option pricing models based upon alternative risk-neutral density function.

Whilst Black-Scholes (1973) model supposes that the continuous underlying asset return is normally distributed, Jarrow-Rudd (1982) have proposed a
method based on statistical series expansions to price options when densities are skewed and leptokurtic. The Black-Scholes (1973) model is then, a special case of the Jarrow-Rudd (1982) model. The unknown state price density of the underlying asset return is approximated by using the information of skewness and kurtosis departures from Gaussianity. In this approach, only the first moments of the risk-neutral distribution are needed and can be approximated using their empirical counterparts estimated on the data.

### 2.2.2 The Jarrow and Rudd (1982) Model

Following Jarrow and Rudd (1982), we assume that the approximate distribution of the asset price $g(S_T)$ is the log-normal distribution $l(S_T)$, with the two first centered moments equal to the “true” ones, that is:

$$
\kappa_1(f) = \kappa_1(l) \quad \text{and} \quad \kappa_2(f) = \kappa_2(l) = [\kappa_1(l)]^2 \left( e^{\sigma^2 T} - 1 \right).
$$

(19)

using a Gram-Charlier series expansion, the risk-neutral density function can be written as:

$$
f(S_T) = l(S_T) - \frac{k_3}{3!} \frac{d^3l(S_T)}{dS_T^3} + \frac{k_4}{4!} \frac{d^4l(S_T)}{dS_T^4} + \varepsilon(S_T)
$$

(20)

where $l(S_T)$ is the log-normal density function, $k_3 = \kappa_3(f) - \kappa_3(l)$, $k_4 = \kappa_4(f) - \kappa_4(l)$ and $\varepsilon(S_T)$ is a residual.

Substituting this expression into the risk-neutral valuation operator (4), yields the following theorem.

**Theorem 1** (Jarrow and Rudd, 1982). Under the hypotheses of existence of the first five non-central moments of the underlying asset terminal price density, the choice of the log-normal as the approximate density of the underlying asset terminal price density and perfection and completeness of financial markets, the fair price of an European call option $C_{JR}$ written on a stock $S_t$ with strike price $K$ is:

$$
C_{JR} = C[S_t, K, \tau, r, \nu, f, l, \sigma, \kappa_3, \kappa_4]
= e^{-r\tau} \int_{S_T = K}^{+\infty} (S_T - K) \left[ l(S_T) - \frac{k_3}{3!} \frac{d^3l(S_T)}{dS_T^3} \right] dS_T + \varepsilon(S_T)
$$

(21)

where $\varepsilon(S_T)$ is a residual.

**Proof**: see previous discussion.

---

20 These restrictions are justified by an heuristic argument of goodness-of-fit of the approximating density to the approximated one. The first restriction is moreover justified theoretically by a risk-neutrality argument (see Jarrow and Rudd, 1982 and Appendix 3), while the second is proposed in order to avoid problems of multicolinearity between second and fourth moment (see Corrado and Su, 1996-a).
Developing equation (21), the Jarrow and Rudd European call option price can be expressed as:

\[
C_{JR} = C_{BS} - e^{-rt} \frac{k_3}{3!} \int_{S_T=K}^{+\infty} (S_T - K) \frac{d^4I(S_T)}{dS_T^4} dS_T
\]
\[
+ e^{-rt} \frac{k_4}{4!} \int_{S_T=K}^{+\infty} (S_T - K) \frac{d^4I(S_T)}{dS_T^4} dS_T + \zeta(S_T)
\]

where \(C_{BS}\) is the price of an European call and \(d\) the standard moneyness measure under the Black and Scholes (1973) hypotheses.

The second term of the equation (22) corrects the pricing error due to the asymmetry of the original distribution function, whilst the third allows to take into account the phenomenon of heavy tails and the fourth term is a residual depending on the strike price. This statistical series expansion could obviously be based on higher moments, but one can think that higher moments than the fourth one, if they exist, would bring no supplementary valuable information. If the risk-neutral density of the underlying asset price is log-normal, then \(k_j = 0\) for \(j = [3, 4]\), and equation (22) collapses to the Black and Scholes (1973) formula.

Recalling that \(\kappa_3(\cdot) = \mu_3(\cdot), \kappa_4(\cdot) = \mu_4(\cdot) - 3\mu_2(\cdot)^2\) and \(\kappa_2(\cdot) = \kappa_2(l)\), it is then possible to obtain the following explicit formula for the price of an European call option.

**Corollary 1** (Corrado and Su, 1996-a). Under the hypotheses of existence of the first five non-central moments of the underlying asset terminal price density, the choice of the log-normal as the approximate density of the underlying asset terminal price density and perfection and completeness of financial markets, the fair price of an European call option \(C_{JR}\) written on a stock \(S_t\) with strike price \(K\) can also be written as:

\[
C_{JR} \simeq C_{BS} + \lambda_1 Q_3 + \lambda_2 Q_4
\]

with:

\[
\begin{cases} 
Q_3 = -(S_t e^{rt})^3 \left(e^{\sigma^2 \tau} - 1\right) \frac{3/2}{\sigma^2} \frac{e^{-tr}}{d - 2\sigma\sqrt{\tau}} \frac{I(K)}{K^2\sigma^2} \\
Q_4 = (S_t e^{rt})^4 \left(e^{\sigma^2 \tau} - 1\right)^2 \frac{4}{d^2 - 5d\sigma\sqrt{\tau} + 6\sigma^2\tau - 1} \frac{I(K)}{K^4\sigma^4}
\end{cases}
\]

and:

\[
\begin{cases} 
\lambda_1 = [\gamma_1(f) - \gamma_1(l)] \\
\lambda_2 = [\gamma_1(f) - \gamma_1(l)]
\end{cases}
\]

where the remainder term \(\zeta(S_T)\) have been neglected in (23); \(\gamma_1(\cdot)\) and \(\gamma_2(\cdot)\)
are the Fisher parameters for skewness and kurtosis:\(^{21}\):

\[
\gamma_1 (\cdot) = \frac{\mu_3 (\cdot)}{\mu_2^{3/2} (\cdot)} \quad \text{and} \quad \gamma_2 (\cdot) = \frac{\mu_4 (\cdot)}{\mu_2^2 (\cdot)} - 3
\]

and \(\mu_i (\cdot)\) are the centered moment of order \(i\), \(i = [2, 3, 4]\).

**Proof:** see Appendix 3.

The coefficients \([\gamma_1 (f) - \gamma_1 (l)]\) and \([\gamma_2 (f) - \gamma_2 (l)]\) measure, respectively, the excess skewness and the excess of excess kurtosis of the true risk-neutral density, and characterize the gap between the distribution function of the underlying asset price and the log-normal one. Parameters \(Q_3\) and \(Q_4\) because they also depend on the exercise price relative to options and the standard deviation of the underlying asset, represent the sensitivities of the price of a specific option to departures from log-normality. The difference between Black-Scholes and Jarrow-Rudd induced option prices is then a non-linear function of the excess moments, the level of the volatility of the market and the specific exercise price of the option considered. Figure 1 of Appendix 11 represents the sensitivities of the option price to the excess moments which represents the value of parameters \(-Q_3\) and \(Q_4\) as a function of the moneyness of the specific option under valuation (see also Corrado and Su, 1996-a).

Simulations done by Jarrow and Rudd (1982) show that their formula constitutes a good approximation of the option price when the underlying asset follows a Brownian process with jumps. Moreover, Jarrow and Rudd (1983) test also their relation for pricing individual stock options with market data, and confirm that the use of third and fourth moments seem to improve in-sample the European call option pricing. The same conclusion has been drawn by Corrado and Su (1996-a, 1997-a) who test the Jarrow-Rudd formula on S&P 500 index options traded on the Chicago Board Option Exchange (CBOE). Using optimization techniques to obtain implicit parameter values in-sample, they conclude to a better fit of Jarrow and Rudd (1982) formula out-of-sample.

2.2.3 The Corrado and Su (1996-b and 1997-b) Model

While the Jarrow and Rudd (1982) model leads to a closed-form for option pricing when densities are skewed and leptokurtic, this approach remains nevertheless induly complex since its expression involves the computation of the log-normal distribution derivatives. Following Madan and Milne (1994), an alternative approach is to work with Hermite polynomials series in which the conditional distribution of the underlying asset price log-return - rather than the price itself - is considered, and a standard normal density is used as the

\[
\begin{align*}
\gamma_1 (l) &= 3 \left(e^{\sigma^2 \tau} - 1\right)^{3/2} + \left(e^{\sigma^2 \tau} - 1\right)^{3/2} \\
\gamma_2 (l) &= 16 \left(e^{\sigma^2 \tau} - 1\right) + 15 \left(e^{\sigma^2 \tau} - 1\right)^2 + 6 \left(e^{\sigma^2 \tau} - 1\right)^3 + \left(e^{\sigma^2 \tau} - 1\right)^4
\end{align*}
\]

\(^{21}\)In the case of the log-normal density, Fisher parameters are equal to:
Let the $\tau$-period log-return of the underlying asset $x_\tau$ has a conditional mean $\mu_\tau$ and a standard deviation $\sigma_\tau$, and define the standardized variable $z$ as:

$$
    z = \log \left( \frac{S_\tau}{S_0} \right) - \mu_\tau
$$

(24)

Using a Gram-Charlier type A series expansion, the risk-neutral density function for $z$ is now:

$$
    f(z) = \varphi(z) + \frac{\kappa_3(f)}{3!} H_3(z) \varphi(z) + \frac{\kappa_4(f)}{4!} H_4(z) \varphi(z) + \varepsilon(z)
$$

(25)

where $\varphi(z)$ is the standard normal density function and the standard normal cumulative density, $\kappa_j(\varphi) = \kappa_j(f)$ for $j = [1, 2]$ and $\kappa_j(\varphi) = 0$ for $j = [3, 4]$, $H_i(z)$ denotes the $i$-th Hermite polynomial.

Substituting (25) into the risk-neutral valuation operator, after the change of variable (17) have been performed in (4), Corrado and Su (1996-b and 1997-b) show that the value for an European call option can be obtained from the following theorem.

Theorem 2 (Corrado and Su, 1996-b and 1997-b). Under the hypotheses of existence of the five first non-central moments of the underlying asset log-return density, the choice of the normal as the approximate density of the continuous compound return density, and perfection and completeness of financial markets, the fair price of an European call option $C_{CS}$ written on a stock $S_t$ with strike price $K$ is (with previous notations):

$$
    C_{CS} = C[S_t, K, \tau, r, vGC; f, \varphi, z, \sigma, \kappa_3, \kappa_4] = e^{-rt} \int_{z = \ln(K/S_t) - \mu_\tau}^{+\infty} \left( S_t e^{\mu_\tau + \sigma \sqrt{\tau} z} - K \right) \left[ 1 + \frac{\kappa_3(f)}{3!} H_3(z) + \frac{\kappa_4(f)}{4!} H_4(z) \right] \varphi(z) \, dz + \varsigma \left( \frac{\ln(S_T/K) - \mu_\tau}{\sigma \sqrt{\tau}} \right)
$$

(26)

where $\varsigma(.)$ is a residual.

Proof: see previous discussion.

Developing this expression, the call option price can be written as:

$$
    C_{CS} = C_{BS} + e^{-rt} \frac{\kappa_3(f)}{3!} \int_{z = -\sigma \sqrt{\tau}}^{+\infty} \left( S_t e^{\mu_\tau + \sigma \sqrt{\tau} z} - K \right) H_3(z) \varphi(z) \, dz + e^{-rt} \frac{\kappa_4(f)}{4!} \int_{z = -\sigma \sqrt{\tau}}^{+\infty} \left( S_t e^{\mu_\tau + \sigma \sqrt{\tau} z} - K \right) H_4(z) \varphi(z) \, dz + \varsigma \left( d - \sigma \sqrt{\tau} \right)
$$

(27)

Hermite polynomials are also used in the semi-nonparametric estimation approaches of Gallant and Nychka (1987), Gallant and Tauchen (1989), Gallant et al. (1990) and Lee and Tse (1991).
where $\zeta(\cdot)$ is a residual.

The second and the third terms of the equation take into account the pricing error due to the skewness and the kurtosis deviations from normality.

After recalling that for a standardized random variable, the standard deviation is one, so that:

$$\kappa_3(f) = \gamma_1(f) \quad \text{and} \quad \kappa_4(f) = \gamma_2(f)$$

where $\gamma_1(f)$ and $\gamma_3(f)$ are the Fisher parameters, and dropping the remainder term $\zeta(\cdot)$, successive integrations by parts yield the following corollary.

**Corollary 3** (Corrado and Su, 1996-b and 1997-b). Under the hypotheses of existence of the first five non-central moments of the underlying asset log-return density, the choice of the normal as the approximate density of the continuous compound return density, and perfection and completeness of financial markets, the fair price of an European call option $C_{CS}$ written on a stock $S_t$ with strike price $K$ can also be written as:

$$C_{CS} \simeq C_{BS} + \gamma_1(f) Q_3 + \gamma_2(f) Q_4$$

with:

$$\begin{cases} Q_3 = \frac{1}{\pi} S_t \sigma \sqrt{\tau} (2\sigma \sqrt{\tau} - d) \varphi(d) \\ Q_4 = \frac{1}{\pi} S_t \sigma \sqrt{\tau} (d^2 - 3d \sigma \sqrt{\tau} - 1) \varphi(d) \end{cases}$$

where the remainder term $\zeta(d - \sigma \sqrt{\tau})$ and terms involving $\sigma^{3+3/2}$ and $\sigma^{4+2}$ in (27) are neglected in (29).

**Proof:** see Appendix 4.

Parameters $Q_3$ and $Q_4$, represents respectively the marginal effect of the non normal log-return skewness and kurtosis on the option price24. We illustrates in Figure 2 of Appendix 11 the sensitivities of the option price to departures from Gaussianity.

Simulations done by Backus et al. (1997) show that the Corrado and Su formula constitutes a good approximation of the option price when the underlying asset follows a jump-diffusion process. Moreover, Corrado and Su (1996-b), Kochard (1999) and Brown and Robinson (1999) test the model by using, respectively, S&P 500 index options traded on the Chicago Board Option Exchange (CBOE), S&P 500 index future options traded on the Chicago Mercantile Exchange (CME) and SPI index future options traded on the Sydney Futures Exchange (SFE). They show that the use of higher moments seems to improve significantly the in-sample option pricing accuracy. Corrado and Su (1997-a) also conclude to a better fit of their formula on an out-of-sample basis.

23 This formula is consistent with the Hermite polynomial option pricing model developed by Madan and Milne (1994).

24 Longstaff (1995) suggests that this general option pricing model includes many other pricing models as special cases. Examples of models that are nested within (29) include the Black and Scholes model (1973), the Merton stochastic interest rate model (1973) and the Merton jump diffusion models (1976).
using actively traded individual equity options on the Chicago Board Option Exchange (CBOE).

While the option pricing model based on Gram-Charlier series expansion leads to analytic expressions for the option price, as it has been pointed previously, the successive terms that appear in the series expansion of the risk-neutral density are not necessarily in decreasing order of importance, so that the expansion may not converge regularly. The use of an Edgeworth series expansion can encompass this problem.

2.2.4 The Rubinstein (1998) Model

Following Rubinstein (1998), we consider a normal Edgeworth series expansion as a natural candidate for approximating the “true” risk-neutral density of the underlying asset log-return. In this case, recalling that the density expansion is:

\[ f(z) = \varphi(z) + \frac{\kappa_3(f)}{3!}H_3(z)\varphi(z) + \left\{ \frac{\kappa_4(f)}{4!}H_4(z) + 10\frac{[\kappa_4(f)]^2}{6!}H_6(z) \right\}\varphi(z) + \varepsilon(z) \quad (30) \]

where \( z \) is defined as in (27), \( \varphi(z) \), \( H_i(z) \) and \( \varepsilon(z) \) denotes respectively the standard normal density, the \( i \)-th Hermite polynomial and a residual.

Applying the same steps as previously, the Edgeworth series expansion based option price can be expressed in the following theorem.

**Theorem 3** (Rubinstein, 1998). Under the hypotheses of existence of the five first non-central moments of the underlying asset log-return density, the choice of the normal as the approximate density of the continuous compound return density, and perfection and completeness of financial markets, the fair price of an European call option \( C_R \) written on a stock \( S_t \) with strike price \( K \) can be written as:

\[ C_R \simeq C[S_t, K, \tau, r, v_E, f, \varphi, z, \sigma, \kappa_3, \kappa_4] \quad (31) \]

\[ = C_{BS} + \gamma_1(f)Q_3'' + \gamma_2(f)Q_4'' + \gamma_1(f)^2Q_5'' \]

where the remainder term \( \eta(d - \sigma\sqrt{\tau}) \) and terms involving \( \sigma^3\tau^{3/2} \) and \( \sigma^4\tau^2 \) in (30) are neglected in (31); \( Q_3'' \) and \( Q_4'' \) are the same as in (29) and:

\[ Q_5'' = \frac{10}{6!}S_t\sigma\sqrt{\tau} (d^4 - 5d^3\sigma\sqrt{\tau} - 6d^2 + 15d\sigma\sqrt{\tau} + 3) \varphi(d) \]

**Proof:** see Appendix 5.

In comparison with the Jarrow-Rudd model, the use of an Edgeworth instead of a Gram-Charlier type A series expansion leads to an extra term - function of the excess skewness, and to a term denoted \( Q_5'' \) in the closed-form of the
fair price of an option. Figure 3 and 4 of Appendix 11 represent, respectively, the effect of skewness on the Rubinstein’s price and the differences between the Black-Scholes and the Jarrow-Rudd, Corrado-Su and Rubinstein call option prices in presence of skewness and kurtosis.

2.3 Implied Probability Densities, Implied Volatility Smile Functions and the Greeks

The following section is dedicated to the implied density functions related to the multi-moment approximate option pricing models. It is indeed possible to express implied probability distribution based on expansions. Once backed out the implied moments from market prices, this would lead for instance to an approximation of agents’ expectations.

2.3.1 Implied Probability Densities

Using the Jarrow and Rudd (1982), the Corrado and Su (1996-b and 1997-b) and the Rubinstein (1998) European call option pricing models, we get the following expression for the implied risk-neutral density.

Theorem 4. When the European call market price is given by the Jarrow and Rudd (1982) formula, the implied risk-neutral density function of the terminal underlying asset price can be written such as:

\[
    f(S_T) \simeq I(S_T) - \frac{\gamma_1(f) - \gamma_1(l)}{3!} (S_t)^3 \left( e^{\sigma^2 T} - 1 \right)^{3/2} \frac{d^3 I(S_T)}{dS_t^3} \tag{32}
\]

where: \( I(S_T) \) is the lognormal density function.

Proof: see previous discussion.

The implied risk-neutral density function can be expressed as a linear function of the excess skewness and excess of excess kurtosis of the underlying asset price.

Theorem 5. When the European call market price is given by the Corrado and Su (1996-b and 1997-b) formula, the implied risk-neutral density function of the continuous compounded asset return can be written as such as:

\[
    f(z) \simeq \varphi(z) \left[ 1 + \frac{\gamma_1(f)}{3!} (z^3 - 3z) + \frac{\gamma_2(f)}{4!} (z^4 - 6z^2 + 3) \right] \tag{33}
\]
where $\varphi(.)$ is the standard normal density function and $z$ is defined as in (17).

**Proof:** see previous discussion.

The implied state price distribution function is then a linear function of the skewness and the excess kurtosis of the underlying asset log-return.

**Theorem 6.** When the European call market price is given by the Rubinstein (1998) formula, the implied risk-neutral density function of the continuous compounded asset return can be written such as:

$$f(z) \approx \varphi(z) \left[ 1 + \frac{\gamma_1(f)}{3!} (z^3 - 3z) + \frac{\gamma_2(f)}{4!} (z^4 - 6z^2 + 3) \right]$$

$$+ \frac{[\gamma_1(f)]^2}{6!} (z^6 - 15z^4 + 45z^2 - 15)$$

where $\varphi(.)$ is the standard normal density function and $z$ is defined as in (17).

**Proof:** see previous discussion.

As with the Gram-Charlier Type A series expansion, the implied risk-neutral density function can be written as a function of the skewness and excess kurtosis of the underlying asset log-return; the relation is no longer linear but quadratic.

In Figure 5 of Appendix 11, we display simultaneously the Black-Scholes and the Jarrow-Rudd, Corrado-Su and Rubinstein probability density functions with significant skewness and kurtosis.

### 2.3.2 Implied Volatility Smile Functions

Following the approach of Backus et al. (1997) and, more recently of Bakshi et al. (2002), we can also provide the implied volatility function $ISD$ that corresponds to a volatility, denoted $\Psi$, that equates the market price of the option to the value given by the Black-Scholes (1973) formula, other values and parameters fixed. Using Jarrow-Rudd (1982), Corrado-Su (1996-b and 1997-b) and Rubinstein (1998) European call option pricing models, we get the following expressions for the implied volatility smile functions$^{25}$.  

**Theorem 7.** When the European call market price is given by the Jarrow and Rudd (1982) formula, the implied volatility function can be written such as$^{26}$:

$$ISD_{JR} = \Psi [S_t, K, \tau, r, v_{GC}, f, l, S_T, \sigma, \kappa_3, \kappa_4]$$

$$\approx \sigma \sqrt{\tau} + \lambda_1 \sigma \sqrt{\tau} Q_3'' \left[ \varphi(d) \right]^{-1}$$

$$+ \lambda_2 \sigma \sqrt{\tau} Q_4'' \left[ \varphi(d) \right]^{-1}$$

$$Q_3'' = -(S_t e^{r\tau})^2 (e^{\sigma^2 \tau} - 1)^{3/2} (d - 2 \sigma \sqrt{\tau}) \frac{\ln(K)}{\sigma \sqrt{\tau}}$$

$$Q_4'' = (S_t e^{r\tau})^3 (e^{\sigma^2 \tau} - 1)^2 (d^2 - 5d \sigma \sqrt{\tau} + 6 \sigma^2 \tau - 1) \frac{\ln(K)}{2 \sigma^4 \tau^{3/2}}$$

$^{25}$Following Backus et al. (1997), we refer to the relation between implied volatility and moneyness as the implied volatility smile function.

$^{26}$Equation (35) is an approximation for several reasons: the Gram-Charlier series expansion and a linear approximation of European call prices in terms of volatility (see Appendix 7).
where the implied volatility function ISD corresponds to the value of $\Psi$ that equates the Jarrow and Rudd (1982) price to the value of the Black and Scholes (1973) formula, given the values of other parameters fixed and $\varphi(.)$ is the standard normal density function.

**Proof:** see Appendix 7.

The implied volatility function is then a linear function of the excess skewness and excess of excess kurtosis of the underlying asset log-return risk-neutral density.

**Theorem 8** (see Backus et al., 1997). When the European call market price is given by the Corrado and Su (1996-b and 1997-b) formula, the implied volatility function can be written such as:

$$ISD_{CS} = \Psi [S_t, K, \tau, r, v_{GC}, f, \varphi, z, \sigma, \kappa_3, \kappa_4]$$

$$\simeq \sigma \sqrt{\tau} + \frac{\gamma_1(f)}{3!} \sigma \sqrt{\tau} (2\sigma \sqrt{\tau} - d)$$

$$+ \frac{\gamma_2(f)}{4!} \sigma \sqrt{\tau} (d^2 - 3 d \sigma \sqrt{\tau} - 1)$$

(36)

where the implied volatility ISD$_{CS}$ corresponds to the value of $\Psi$ that equates the Corrado and Su (1996-b and 1997-b) price to the value of the Black and Scholes (1973) formula, given the values of other parameters fixed.

**Proof:** see Appendix 7.

The implied volatility function can again be expressed as a linear function of the skewness and excess kurtosis of the underlying asset log-return risk-neutral density.

**Theorem 9.** When the European call market price is given by the Rubinstein (1998) formula, the implied volatility function reads:

$$ISD_R = \Psi [S_t, K, \tau, r, v_E, f, \varphi, z, \sigma, \kappa_3, \kappa_4]$$

$$\simeq \sigma \sqrt{\tau} + \frac{\gamma_1(f)}{3!} \sigma \sqrt{\tau} (2\sigma \sqrt{\tau} - d) + \frac{\gamma_2(f)}{4!} \sigma \sqrt{\tau} (d^2 - 3 d \sigma \sqrt{\tau} - 1)$$

$$+ 10 \frac{[\gamma_1(f)]^2}{6!} \sigma \sqrt{\tau} [d^4 - 5 d^3 \sigma \sqrt{\tau} - 6 d^2 + 15 d \sigma \sqrt{\tau} + 3]$$

(37)

where the implied volatility ISD$_R$ corresponds to the value of $\Psi$ that equates the Rubinstein price (1998) to the value of the Black and Scholes (1973) formula, given the values of other parameters fixed.

**Proof:** see Appendix 7.

As with the Gram-Charlier Type A series expansion, the implied volatility function can here be expressed as a function of the skewness and excess kurtosis

---

27 Equation (36) is an approximation for several reasons: the Gram-Charlier series expansion, a linear approximation of European call prices in terms of volatility and the elimination of terms involving $\sigma^2 \tau^{3/2}$ and $\sigma^4 \tau^2$ (see Appendix 4 and 7).

28 Using the same approach, Baschi et al. (2002) derive a similar relation between the implied volatility and the skewness and the kurtosis of the risk-neutral distribution. The only difference is that they do not identified explicitly the coefficients of implied volatility function.
of the underlying asset log-return risk-neutral density; the relation is no longer linear but quadratic as in the previous case.

Figure 6 illustrates the comparison of Jarrow-Rudd, Corrado-Su and Rubinstein’s implied volatility smile functions when the risk-neutral density is skewed and leptokurtic. Figures 7 to 9 are dedicated to the comparison of the specific effect of the skewness and of the kurtosis on the shape of the implied volatility smile functions.

2.3.3 The Greeks

The Greek parameters are of interests since they can be used for testing and hedging purposes. In particular, Delta states the sensitivity of the option price to underlying asset price movements. By definition, it is the first partial derivative of the option price with respect to the underlying asset price. Gamma measures the sensitivity of Delta-hedged strategies to the underlying asset price changes and is defined by the second partial derivative of the option price with respect to the underlying asset price. Vega, Khi and Psi measure the sensitivities of the option price with respect to changes in the volatility, skewness and kurtosis and are defined by the first partial derivatives of the option price. By taking the appropriate derivatives of the Jarrow-Rudd (1982), the Corrado-Su (1996-b and 1997-b) and the Rubinstein (1998) European call option pricing models, we get the following expressions for the Delta, Gamma, Khi and Psi.

**Theorem 10** (see Corrado and Su, 1996-b and 1997-b). When the European call market price is given by the Jarrow and Rudd (1982) formula, the Delta, Gamma, Vega, Khi and Psi of the call can be written respectively such as:

\[
\Delta_{JR}^C = \frac{\partial C_{JR}}{\partial S_t} \\
\cong \Phi(d) + \lambda_1 Q_3''' \left( -d^2 + 6d \sigma \sqrt{\tau} - 8\sigma^2 \tau + 1 \right) \\
+ \lambda_2 Q_4''' \left( -d^3 + 10d^2 \sigma \sqrt{\tau} - 31d\sigma^2 \tau + 3d + 30\sigma^3 \tau^{3/2} - 10\sigma \sqrt{\tau} \right) \\
\]  

\[
\Gamma_{JR}^C = \frac{\partial^2 C_{JR}}{\partial S_t^2} \\
\cong \left( S_t \sigma \sqrt{\tau} \right)^{-1} \left\{ \varphi(d) + \lambda_1 Q_3'''' \left( d^3 - 24\sigma^2 \tau^{3/2} - 9d^2 \sigma \sqrt{\tau} \right) \\
+ 26d \sigma^2 \tau - 3d + 9\sigma \sqrt{\tau} \right) + \lambda_2 Q_4'''' \left( d^4 - 14d^3 \sigma \sqrt{\tau} + 71d^2 \sigma^2 \tau \\
- 6d^2 - 154d \sigma^3 \tau^{3/2} + 42d \sigma \sqrt{\tau} + 120 \sigma^4 \tau^2 - 71 \sigma^2 \tau + 3 \right) \right\} \\
\]
\begin{align*}
v_{JR}^C &= \frac{\partial C_{JR}}{\partial \sigma} \\
&\approx S_1 \sqrt{T} \left\{ \varphi(d) + \lambda_1 Q'''_3 (d^3 - 3d^2 \sigma \sqrt{T} - 3d + 2d \sigma^2 \tau + 3 \sigma \sqrt{T}) \\
&\quad + \lambda_2 Q''_4 (d^4 - 6d^3 \sigma \sqrt{T} + 11d^2 \sigma^2 \tau - 6d \sigma^3 \tau^{3/2} + 18d \sigma \sqrt{T} \\
&\quad - 11 \sigma^2 \tau - 6d^2 + 3) \\
&\quad + 3d^2 \tau e^{\sigma^2 \tau} Q''_3 \left[ \lambda_1 \left( e^{\sigma^2 \tau} - 1 \right)^{-1} - \left( e^{\sigma^2 \tau} - 1 \right)^{-1/2} - \left( e^{\sigma^2 \tau} - 1 \right)^{-1/2} \right] \\
&\quad + 4e^{\sigma^2 \tau} Q''_4 \left[ \lambda_2 \left( e^{\sigma^2 \tau} - 1 \right)^{-1} - 8 - 15 \left( e^{\sigma^2 \tau} - 1 \right) - 9 \left( e^{\sigma^2 \tau} - 1 \right)^2 \\
&\quad - 2 \left( e^{\sigma^2 \tau} - 1 \right)^3 \right] \} \\
&\quad + \psi_{JR}^C = \frac{\partial C_{JR}}{\partial \gamma_1 (f)} \approx Q_3 \\
\chi_{JR}^C &= \frac{\partial C_{JR}}{\partial \gamma_2 (f)} \approx Q_4 \\
\end{align*}

and:

\begin{align*}
\Psi_{JR}^C &= \frac{\partial C_{JR}}{\partial \gamma_2 (f)} \approx Q_4
\end{align*}

with:

\begin{align*}
Q'''_3 &= - (S_t e^{\tau})^2 \left( e^{\sigma^2 \tau} - 1 \right)^{-3/2} \frac{l(K)}{3K \sigma \tau^2} \\
Q''_4 &= (S_t e^{\tau})^3 \left( e^{\sigma^2 \tau} - 1 \right)^{-2} \frac{l(K)}{4K \sigma^3 \tau^{3/2}}
\end{align*}

where \( \Phi(.) \), \( \varphi(.) \) and \( \psi(.) \) are, respectively, the cumulative density function of the standard Gaussian distribution, the density function of the standard Gaussian distribution and the density function of the lognormal distribution and \( \lambda_1, \lambda_2, d, Q_3, Q_4, Q'''_3 \) and \( Q''_4 \), are defined in equation (23) and (35).

**Proof:** see Appendix 8.

**Theorem 11** (see Backus et al., 1997). When the European call market price is given by the Corrado and Su (1996-b and 1997-b) formula, the Delta, Gamma, Vega, Khi and Psi of the call can be written respectively such as:

\begin{align*}
\Delta_{CS} &= \frac{\partial C_{CS}}{\partial S_t} \\
&\approx \Phi(d) + \varphi(d) \left\{ \frac{\gamma_1(f)}{3!} \left( d^2 - 3d \sigma \sqrt{T} + 2 \sigma^2 \tau - 1 \right) \\
&\quad + \frac{\gamma_2(f)}{4!} \left[ -d^3 + 4d^2 \sigma \sqrt{T} + 3d - 3d \sigma^2 \tau - 4 \sigma \sqrt{T} \right] \right\}
\end{align*}
\[ \Gamma_{CS}^{C} = \frac{\partial^2 C_{CS}}{\partial S_t^2} \]  

\[ \simeq \varphi(d) \left\{ 1 + \frac{\gamma_1(f)}{3!} (-d^3 + 3d^2\sigma\sqrt{T} - 2d\sigma^2\tau + 3d - 3\sigma\sqrt{T}) ight. 
\]  

\[ \left. + \frac{\gamma_2(f)}{4!} (d^4 - 4d^3\sigma\sqrt{T} + 3d^2\sigma^2\tau - 6d^2 + 12d\sigma\sqrt{T} - 3\sigma^2\tau + 3) \right\} \]  

\[ \nu_{CS}^{C} = \frac{\partial C_{CS}}{\partial \sigma} \]  

\[ \simeq \varphi(d) S_t\sqrt{T} \left\{ 1 + \frac{\gamma_1(f)}{3!} (-d^3 + 3d^2\sigma\sqrt{T} - 2d\sigma^2\tau + 3\sigma\sqrt{T}) \right. \]  

\[ \left. + \frac{\gamma_2(f)}{4!} (d^4 - 4d^3\sigma\sqrt{T} + 3d^2\sigma^2\tau - 2d^2 - 3\sigma^2\tau - 1) \right\} \]  

\[ \chi_{CS}^{C} = \frac{\partial C_{CS}}{\partial \gamma_1(f)} \simeq Q_3' \]  

\[ \Psi_{CS}^{C} = \frac{\partial C_{CS}}{\partial \gamma_2(f)} \simeq Q_4' \]  

where \( \Phi(.) \) and \( \varphi(.) \) are the cumulative density function and the density function of the standard Gaussian distribution and \( d \), \( Q_3' \) and \( Q_4' \) are defined respectively in equation (18) and (29).

**Proof:** see Appendix 9.

**Theorem 12.** When the European call market price is given by the Rubinstein (1998) formula, the Delta, Gamma, Khi and Psi of the call can be written respectively such as:

\[ \Delta_{R}^{C} = \frac{\partial C_{R}}{\partial S_t} \]  

\[ \simeq \Phi(d) + \varphi (d) \left\{ \frac{\gamma_1(f)}{3!} (d^2 - 3d\sigma\sqrt{T} + 2\sigma^2\tau - 1) \right. \]  

\[ \left. + \frac{\gamma_2(f)}{4!} [-d^3 + 4d^2\sigma\sqrt{T} + 3d (1 - \sigma^2\tau) - 4\sigma\sqrt{T}] \right. \]  

\[ \left. + \frac{10}{6!} [\gamma_1(f)]^2 [-d^5 + 6d^4\sigma\sqrt{T} - 5d^3\sigma^2\tau + 10d^3 - 36d^2\sigma\sqrt{T} - 36\sigma^2\tau - 4d + 18\sigma\sqrt{T}] \right\} \]
\[ \Gamma^C_R = \frac{\partial^2 C_R}{\partial S^2_I} \quad (49) \]
\[ \simeq \frac{\varphi(d)}{S_I \sigma \sqrt{T}} \left\{ 1 + \frac{\gamma_1(f)}{3!} \left( -d^3 + 3d^2 \sigma \sqrt{T} - 2d \sigma^2 \tau + 3d - 3 \sigma \sqrt{T} \right) \right. \\
+ \frac{\gamma_2(f)}{4!} \left( d^4 - 4d^3 \sigma \sqrt{T} + 3d^2 \sigma^2 \tau - 6d^2 + 12d \sigma \sqrt{T} - 3 \sigma^2 \tau + 3 \right) \right. \\
+ \frac{10 \gamma_1(f)^2}{6!} \left( d^6 - 6d^5 \sigma \sqrt{T} + 5d^4 \sigma^2 \tau - 15d^4 + 60d^3 \sigma \sqrt{T} \right. \\
\left. \left. - 30d^2 \sigma^2 \tau + 45d^2 - 90d \sigma \sqrt{T} + 15 \sigma^2 \tau - 15 \right) \right\} \]
\[ \nu^C_R = \frac{\partial C_R}{\partial \sigma} \quad (50) \]
\[ \simeq \frac{\varphi(d)}{S_I \sigma \sqrt{T}} \left\{ 1 + \frac{\gamma_1(f)}{3!} \left( -d^3 + 3d^2 \sigma \sqrt{T} - 2d \sigma^2 \tau + 3d - 3 \sigma \sqrt{T} \right) \right. \\
+ \frac{\gamma_2(f)}{4!} \left( d^4 - 4d^3 \sigma \sqrt{T} + 3d^2 \sigma^2 \tau - 6d^2 + 2d^2 - 3 \sigma^2 \tau - 1 \right) \right. \\
+ \frac{10 \gamma_1(f)^2}{6!} \left( d^6 - 6d^5 \sigma \sqrt{T} + 5d^4 \sigma^2 \tau - 15d^4 + 30d^3 \sigma \sqrt{T} \right. \\
\left. \left. - 30d^2 \sigma^2 \tau + 9d^2 + 15 \sigma^2 \tau + 3 \right) \right\} \]
\[ \chi^C_R = \frac{\partial C_R}{\partial \gamma_1(f)} \simeq Q^j_3 + 2 \gamma_1(f) Q^j_5 \quad (51) \]
and:
\[ \Psi^C_R = \frac{\partial C_R}{\partial \gamma_2(f)} \simeq Q^j_4 \quad (52) \]
where \( \Phi(.) \) and \( \varphi(.) \) are, respectively, the cumulative density function and the density function of the standard Gaussian distribution and \( d, Q^j_3, Q^j_4 \) and \( Q^j_5 \) are defined respectively in equation (18), (29) and (31).

**Proof:** see Appendix 10.

The first terms on the right-hand sides of equations (38), (43), (48), (39), (44), (49), (40), (45) and (50) are respectively the Delta, Gamma and Vega of the Black-Scholes (1973) model whilst second terms adjust the Delta, Gamma and Vega for the presence of skewness and kurtosis in the return distribution. In Figure 10 to 12, we illustrate respectively the differences in Deltas, Gammas and Vegas for the approximate models. Figure 13 displays the Khi of the Corrado-Su and Rubinstein models, and Figure 14 represents the comparison between related Psi.

### 3 Conclusion

This article focuses on a way of encompassing drawbacks of Black and Scholes (1973) model using statistical series expansions to correct the implied density..
departures from Gaussianity. We investigate several different multi-moment approximate option pricing models in an unified framework, highlighting the difference between Jarrow and Rudd (1982), Corrado and Su (1996-b and 1997-b) and Rubinstein (1998) models. We present the conditions that ensure the respect of the martingale restriction and establish the link between these approximate models and alternative option pricing models such as Black and Scholes (1973) and Hermite polynomial models (see Madan and Milne, 1994, Abken et al., 1996). We also provide analytical formulae for related implied densities and implicit volatility smile functions, and illustrate their properties with simulated data. The final contribution of this paper concerns hedging parameters in this setting: we extend the traditional Greeks to deal with higher moment changes.

Our next work will consist in investigating the relative pricing power and hedging performances of all these different fourth-moment options pricing models with market data.

References


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Appendix 1

It is possible to express - under mild conditions, see below - a continuous density \( f(x) \) as a function of an arbitrary continuous density \( g(x) \) and its cumulants \( \kappa_j(\cdot), j = [1, ..., N] \), as follows:

\[
 f(x) = g(x) + \sum_{i=0}^{+\infty} \frac{1}{i!} \left\{ \sum_{j=1}^{N-1} (-1)^j \frac{\kappa_j(f) - \kappa_j(g)}{j!} D^j \right\}^i g(x) + \varepsilon(x) \tag{6}
\]

**Proof.** Let \( F(x) \) and \( G(x) \) be respectively the “true” cumulative density function and the approximating one. We assume moreover that \( dF(x)/dx = f(x) \) and \( dG(x)/dx = g(x) \) exist, as well as the first \( N \) non central moments of the distribution function \( F \). Formally, the first cumulants \( \kappa_j(f), j = [1, ..., N - 1] \), are given by the following equality (see Kendall and Stuart, 1977, p.73):

\[
 \ln \phi(f, t) = \left[ \sum_{j=1}^{N-1} \kappa_j(f) \frac{(it)^j}{j!} \right] + o(t^{N-1}) \tag{A.1.1.}
\]

where \( \phi(f, t) \) is the characteristic function of \( f(x) \) and \( i^2 = -1 \).

Taking exponential of equation (A.1.1.) and using the definition of the characteristic function of \( g(x) \) we obtain:

\[
 \phi(f, t) = \exp \left[ \sum_{j=1}^{N-1} [\kappa_j(f) - \kappa_j(g)] \frac{(it)^j}{j!} \right] \phi(g, t) + o(t^{N-1}) \tag{A.1.2.}
\]

Taking the inverse Fourier transform of (A.1.2.), yields (see Johnson et al., 1994, p.26):

\[
 f(x) = \exp \left\{ \sum_{j=1}^{N-1} (-1)^j \frac{k_j(D)^j}{j!} \right\} g(x) + \varepsilon(x) \tag{A.1.3.}
\]

with:

\[
 \begin{align*}
 f(x) & = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(f, t) \, dt \\
 g(x) & = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(g, t) \, dt \\
 \exp \left[ (-1)^j D^j \right] g(x) & = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \exp \left[ k_j (it)^j \right] \phi(g, t) \, dt \\
 \varepsilon(x) & = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} o(t^{N-1}) \, dt
\end{align*}
\]

where \( k_j = \kappa_j(f) - \kappa_j(g) \) and \( D \) is the differentiation operator such as \( D^j g(x) = d^j g(x)/dx^j \).

Expanding the equation (A.1.3.) as an infinite polynomial leads to the desired result, that is:

\[
 f(x) = \sum_{i=0}^{+\infty} \frac{1}{i!} \left\{ \sum_{j=1}^{N-1} (-1)^j \frac{k_j(D)^j}{j!} \right\}^i g(x) + \varepsilon(x) \tag{A.1.4.}
\]

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Appendix 2

The $j$-th cumulant for a normalized sum $x$ of $n$ independent and identically standardized random variables $x_i$, that is:

$$x = \sigma^{-1} \sum_{i=1}^{n} (x_i - \mu)$$  \hspace{1cm} (9)

is proportional to $n^{1-j/2} (j \geq 2)$.

**Proof.** By construction, the characteristic function of $x$ must verify the following equality:

$$\phi(t) = E \left[ e^{itx} \right] = \left[ \phi_i \left( t/n^{1/2} \right) \right]^n$$  \hspace{1cm} (A.2.1.)

where $-\infty < t < +\infty, i^2 = -1$ and $\phi_i(t)$ is the characteristic function of $\sigma^{-1} (x_i - \mu)$.

Recalling the definition of cumulants (A.1.1.), we must also have, for $x$:

$$\phi(t) = \left\{ \frac{1}{2} (it)^2 + \frac{1}{3!} \kappa_3 (it)^3 + \frac{1}{4!} \kappa_4 (it) \right\} + o(t^4)$$  \hspace{1cm} (A.2.2.)

where $\kappa_j, j = [3, 4]$, refers to the $j$-th cumulant of $x$, with $\kappa_1 = 0$ and $\kappa_2 = 1$.

Following the same approach for $\sigma^{-1} (x_i - \mu)$, we get:

$$\phi_i(t) = \left\{ \frac{1}{2} (it)^2 + \frac{1}{3!} \kappa_{i,3} (it)^3 + \frac{1}{4!} \kappa_{i,4} (it) \right\} + o(t^4)$$  \hspace{1cm} (A.2.3.)

where $\kappa_{i,j}, (i \times j) = (1, ..., n) \times (3, 4)$, refers now to the $j$-th cumulant of the standardized random variable $\sigma^{-1} (x_i - \mu)$ with $k_{i,1} = 0$ and $k_{i,2} = 1$.

Using equation (A.2.1.) and equation (A.2.3.), we have:

$$\phi(t) = \left\{ -\frac{1}{2} t^2 + n^{-1/2} \frac{1}{3!} \kappa_{i,3} (it)^3 + n^{-1} \frac{1}{4!} \kappa_{i,4} (it) \right\} + o(t^4)$$  \hspace{1cm} (A.2.4.)

Identifying terms in (A.2.2.) and in (A.2.4.) leads to the desired property, that is:

$$\kappa_j = n^{1-j/2} \kappa_{i,j}$$  \hspace{1cm} (A.2.5.)

with $\kappa_j$ the $j$-th cumulant of $x$, $j \geq 2$ and $\kappa_{i,j}$ the $j$-th cumulant of $\sigma^{-1} (x_i - \mu)$.  

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Appendix 3

Under the hypotheses of existence of the five first non-central moments of the underlying asset terminal price density, the choice of the log-normal as the approximate density of the underlying asset terminal price density and perfection and completeness of financial markets, the fair price of an European call $C_{JR}$ can be expressed as:

$$C_{JR} \simeq C_{BS} + \lambda_1 Q_3 + \lambda_2 Q_4$$

with:

$$
\begin{align*}
Q_3 &= - (S_t e^{rT})^3 \left( e^{\sigma^2 T} - 1 \right)^{3/2} \frac{e^{-rT}}{K \sigma \sqrt{T}} \frac{\mu(K)}{K} \\
Q_4 &= (S_t e^{rT})^4 \left( e^{\sigma^2 T} - 1 \right)^2 \frac{e^{-2rT}}{4T} \left( d^2 - 5d \sigma \sqrt{T} + 6 \sigma^2 T - 1 \right) \frac{\mu(K)}{K^2 \sigma^2 T}
\end{align*}
$$

and:

$$
\begin{align*}
\lambda_1 &= \left[ \gamma_1 (f) - \gamma_1 (0) \right] \\
\lambda_2 &= \left[ \gamma_2 (f) - \gamma_2 (0) \right]
\end{align*}
$$

Proof. Under a lognormal Gram-Charlier series expansion, the risk-neutral price of an European call written on a stock $S_t$ with strike price $K$ is:

$$
C_{JR} = C [S_t, K, \tau, r, \mu, \sigma, \nu_3, \nu_4, f, l, z, \kappa_3, \kappa_4] \\
= e^{-rT} \int_{S_T = K}^{+\infty} (S_T - K) \left[ l(S_T) - \frac{k_3 d^3 l(S_T)}{3!} \frac{dS_T^3}{S_T^3} + \frac{k_4 d^4 l(S_T)}{4!} \frac{dS_T^4}{S_T^4} \right] dS_T
$$

(A.3.1.)

where the Gram-Charlier series expansion residual is dropped.

In order to evaluate expression (A.3.1.), we need to calculate the following integral, for $j = [3, 4]$:

$$I_j = \int_{S_T = K}^{+\infty} (S_T - K) \frac{d^j l(S_T)}{dS_T^j} dS_T$$

(A.3.2.)

Integrating equation (A.3.2.) by parts, gives:

$$I_j = \begin{cases} 
\left. (S_T - K) \frac{d^{j-1} l(S_T)}{dS_T^{j-1}} \right|_K^{+\infty} - \left[ \frac{d^{j-2} l(S_T)}{dS_T^{j-2}} \right]_K^{+\infty}, & j = 3 \\
\lim_{S_T \to +\infty} S_T \frac{d^{j-1} l(S_T)}{dS_T^{j-1}} - \lim_{S_T \to +\infty} K \frac{d^{j-1} l(S_T)}{dS_T^{j-1}}, & j = 4
\end{cases}$$

(A.3.3.)
Using the following property of the lognormal distribution for \( k \geq 0 \) (see Kendall, 1977, p.180):

\[
\lim_{S_T \to +\infty} (S_T)^k l (S_T) = 0 \tag{A.3.4.}
\]

Integral \( (A.3.2.) \) then becomes, for \( j = [3, 4] \):

\[
I_j = \int_{S_T=K}^{+\infty} (S_T - K) \frac{d^j l (S_T)}{dS_T^j} dS_T = \frac{d^{j-2} l (K)}{dS_T^{j-2}} \tag{A.3.5.}
\]

From the martingale restriction, that is:

\[
S_t = e^{-rT} E_Q [S_T | S_t]
\]

we can derive the first moment of the underlying asset price under the risk-neutral density:

\[
S_t = e^{-rT} \int_0^{+\infty} S_T f (S_T) dS_T
= e^{-rT} \int_0^{+\infty} S_T \left[ l (S_T) - \frac{k_3}{3!} \frac{d^3 l (S_T)}{dS_T^3} + \frac{k_4}{4!} \frac{d^4 l (S_T)}{dS_T^4} \right] dS_T
= e^{-rT} \int_{-\infty}^{+\infty} S_T l (S_T) dS_T
- \frac{k_3}{3!} e^{-rT} \int_0^{+\infty} S_T \frac{d^3 l (S_T)}{dS_T^3} dS_T
+ \frac{k_4}{4!} e^{-rT} \int_0^{+\infty} S_T \frac{d^4 l (S_T)}{dS_T^4} dS_T \tag{A.3.7.}
\]

In order to evaluate expression \( (A.3.7.) \), we need to calculate the following integral for \( j = [3, 4] \):

\[
I_j^* = \int_0^{+\infty} S_T \frac{d^j l (S_T)}{dS_T^j} dS_T \tag{A.3.8.}
\]

Integrating by parts this expression, yields:

\[
I_j^* = \left[ S_T \frac{d^{j+1} l (S_T)}{dS_T^{j+1}} \right]_0^{+\infty} - \left[ \frac{d^j l (S_T)}{dS_T^j} \right]_0^{+\infty}
- \lim_{s_T \to +\infty} S_T \frac{d^{j-1} l (S_T)}{dS_T^{j-1}} - \lim_{s_T \to 0} S_T \frac{d^{j-1} l (S_T)}{dS_T^{j-1}} \tag{A.3.9.}
\]

For the lognormal distribution, we also have for \( j \geq 1 \) (see Kendall, 1977, p.180):

\[
\lim_{s_T \to +\infty} \frac{d^{j-1} l (S_T)}{dS_T^{j-1}} = \lim_{s_T \to 0} \frac{d^{j-1} l (S_T)}{dS_T^{j-1}} = 0 \tag{A.3.10.}
\]
So, using the above expression for \( j = [3, 4] \) in (A.3.7.) and dividing it by \( S_T \), we get the following expression:

\[
1 = e^{-r\tau} \int_{-\infty}^{+\infty} S_T l(S_T) dS_T \]

\[
= e^{-r\tau} \int_{-\infty}^{+\infty} \left( e^{\mu\tau + \sigma\tau z} \right) \varphi(z) \, dz \tag{A.3.11.}
\]

where the change of variable \( z = \ln \left( \frac{S_T}{S_T} - \mu\tau \right) / \sigma\sqrt{\tau} \) have been performed on \( S_T \).

Taking the logarithm of expression (A.3.11.) and rearranging terms, yields:

\[
\mu\tau = r\tau - \frac{1}{2} \sigma^2 \tau \tag{A.3.12.}
\]

Using this expression, the risk-neutral expected value and variance of the terminal price can be written such as:

\[
\begin{align*}
1 &= e^{-r\tau} \int_{-\infty}^{+\infty} S_T l(S_T) dS_T \\
&= e^{-r\tau} \int_{-\infty}^{+\infty} \left( e^{\mu\tau + \sigma\tau z} \right) \varphi(z) \, dz \\
&= e^{(-r\tau + \mu\tau + \frac{1}{2} \sigma^2 \tau)}
\end{align*}
\]

where the change of variable \( z = \ln \left( \frac{S_T}{S_T} - \mu\tau \right) / \sigma\sqrt{\tau} \) have been performed on \( S_T \).

Taking the logarithm of expression (A.3.11.) and rearranging terms, yields:

\[
\mu\tau = r\tau - \frac{1}{2} \sigma^2 \tau
\]

Using this expression, the risk-neutral expected value and variance of the terminal price can be written such as:

\[
\begin{align*}
k_1 (f) &= k_1 (l) = S_t e^{\mu\tau + \frac{1}{2} \sigma^2 \tau} = S_t e^{r\tau} \\
k_2 (f) &= k_2 (l) = [k_1 (l)]^2 \left[ e^{\sigma^2 \tau} - 1 \right] = (S_t e^{r\tau})^2 \left[ e^{\sigma^2 \tau} - 1 \right]
\end{align*}
\]

where the definition of the moments of the lognormal and the equality of the two first cumulants between the true and the approximating distribution have been used.

Substituting (A.3.13.) in equation (A.3.1.) and using the cumulants and the Fisher parameters definitions; \( \kappa_3 (\cdot) = \mu_3 (\cdot), \kappa_4 (\cdot) = \mu_4 (\cdot) - 3\mu_2 (\cdot)^2, \gamma_1 (\cdot) = \mu_3 (\cdot) / \mu_2 (\cdot)^{3/2} \) and \( \gamma_2 (\cdot) = \mu_4 (\cdot) / \mu_2 (\cdot)^2 - 3 \), the value of an European call becomes:

\[
C_{JR} \simeq C_{BS} - [\gamma_1 (f) - \gamma_1 (l)] (S_t e^{r\tau})^3
\]

\[
\times \left( e^{\sigma^2 \tau} - 1 \right)^{3/2} \frac{e^{-r\tau}}{3!} \left( d - 2 \sigma \sqrt{\tau} \right) \frac{l(K)}{K \sigma \sqrt{\tau}} \\
+ [\gamma_2 (f) - \gamma_2 (l)] (S_t e^{r\tau})^4 \left( e^{\sigma^2 \tau} - 1 \right)^2 \frac{e^{-r\tau}}{4!} \\
\times (d^2 - 5d \sigma \sqrt{\tau} + 6 \sigma^2 \tau - 1) \frac{l(K)}{K^2 \sigma^2 \tau}
\]

where \( d \) is defined as in Black and Scholes (1973) formula.

Appendix 4

Under the hypotheses of existence of the five first non-central moments of the underlying asset log-return density, the choice of a normal as the approximate density of the continuous compound return density and perfection and
completeness of financial markets, the fair price of an European call \( C_{CS} \) can be expressed as:

\[
C_{CS} = C_{BS} + \gamma_1(f) Q'_3 + \gamma_2(f) Q'_4
\]  

(26)

with:

\[
\begin{align*}
Q'_3 &= \frac{1}{2} S_t \sigma \sqrt{T} (2 \sigma \sqrt{T} - d) \varphi(d) \\
Q'_4 &= \frac{1}{2} S_t \sigma \sqrt{T} (d^2 - 3d \sigma \sqrt{T} - 1) \varphi(d)
\end{align*}
\]

where terms involving \( \sigma^{3/2} \) and \( \sigma^4 \) are neglected.

**Proof.** Under a Gram-Charlier Type A series expansion, the risk-neutral price of an European call written on a stock \( S_t \) with strike price \( K \) is:

\[
C_{CS} = e^{-rt} \int_{z = \ln(S_T/K)}^{+\infty} (S_T - K) f(z) \, dz
\]

(A.4.1.)

\[
= e^{-rt} \int_{z = \ln(S_T/K)}^{+\infty} S_t e^{\mu_T + \sigma \sqrt{T} z} - K \\
\left[ 1 + \frac{\kappa_3(f)}{3!} H_3(z) + \frac{\kappa_4(f)}{4!} H_4(z) \right] \varphi(z) \, dz
\]

where the change of variable \( z = [\ln(S_T/K) - \mu_T] / \sigma \sqrt{T} \) have been performed on \( S_T \) and the Gram-Charlier expansion residual have been dropped.

In order to evaluate expression (A.4.1.), we need to compute the following integral:

\[
I^*_j = \int_{-d+\sigma \sqrt{T}}^{+\infty} (S_t e^{\mu_T + \sigma \sqrt{T} z} - K) H_j(z) \varphi(z) \, dz
\]

(A.4.2.)

for \( j = [3, 4] \).

Using the definition of Hermite polynomials, we get:

\[
I^*_j = \int_{-d+\sigma \sqrt{T}}^{+\infty} (S_t e^{\mu_T + \sigma \sqrt{T} z} - K) (-1)^j \frac{d^j \varphi(z)}{dz^j} \, dz
\]

(A.4.3.)

\[
= - \int_{-d+\sigma \sqrt{T}}^{+\infty} (S_t e^{\mu_T + \sigma \sqrt{T} z} - K) \frac{d}{dz} \left[ (-1)^{j-1} \frac{d^{j-1} \varphi(z)}{dz^{j-1}} \right] \, dz
\]

\[
= - \int_{-d+\sigma \sqrt{T}}^{+\infty} (S_t e^{\mu_T + \sigma \sqrt{T} z} - K) \frac{d}{dz} H_{j-1}(z) \varphi(z) \, dz
\]

and an integration by parts yields:

\[
I^*_j = - \left[ (S_t e^{\mu_T + \sigma \sqrt{T} z} - K) H_{j-1}(z) \varphi(z) \right]_{-d+\sigma \sqrt{T}}^{+\infty}
\]

(A.4.4.)

\[
+ \sigma \sqrt{T} S_t \int_{-d+\sigma \sqrt{T}}^{+\infty} e^{\mu_T + \sigma \sqrt{T} z} H_{j-1}(z) \varphi(z) \, dz
\]
It is readily verified that the first term in the above expression equals zero. Noting also that \( \lim_{z \to \infty} \varphi(z) = 0 \), this then leaves the expression:

\[
I_j^{**} = \sigma \sqrt{\tau} S_t \int_{-d+\sigma \sqrt{\tau}}^{+\infty} e^{\mu \tau + \sigma \sqrt{\tau} z} H_{j-1}(z) \varphi(z) \, dz \quad (\text{A.4.5.})
\]

Using once again the definition of Hermite polynomials, we have:

\[
I_j^{**} = \sigma \sqrt{\tau} S_t \int_{-d+\sigma \sqrt{\tau}}^{+\infty} e^{\mu \tau + \sigma \sqrt{\tau} z} (\sigma \tau)^{j-1} \frac{d^{j-1} \varphi(z)}{dz^{j-1}} \, dz \quad (\text{A.4.6.})
\]

and integrating by parts, we get:

\[
I_j^{**} = -\sigma \sqrt{\tau} S_t \left[ e^{\mu \tau + \sigma \sqrt{\tau} z} H_{j-2}(z) \varphi(z) \right]_{-d+\sigma \sqrt{\tau}}^{+\infty} + (\sigma \sqrt{\tau})^2 S_t \int_{-d+\sigma \sqrt{\tau}}^{+\infty} e^{\mu \tau + \sigma \sqrt{\tau} z} H_{j-2}(z) \varphi(z) \, dz
\]

Then, by induction, we obtain:

\[
I_j^{**} = \sigma \sqrt{\tau} K H_{j-2}(-d+\sigma \sqrt{\tau}) \varphi(d-\sigma \sqrt{\tau}) + \sigma \sqrt{\tau} \left[ \sigma \sqrt{\tau} K H_{j-3}(-d+\sigma \sqrt{\tau}) \varphi(d-\sigma \sqrt{\tau}) \right] + (\sigma \sqrt{\tau})^2 S_t \int_{-d+\sigma \sqrt{\tau}}^{+\infty} e^{\mu \tau + \sigma \sqrt{\tau} z} H_{j-3}(z) \varphi(z) \, dz
\]

Using the following equality (see Appendix 6 and Stoll and Whaley, 1993, p.245):

\[
K \varphi(d-\sigma \sqrt{\tau}) = S_t e^{\mu \tau + \sigma^2 \tau/2} \varphi(d) \quad (\text{A.4.9.})
\]
leads to the following expression for $I_j^*$:

$$I_j^* = S_t e^{\mu_T + \sigma^2 T/2} \left[ \sum_{k=1}^{\infty} (\sigma \sqrt{T})^k H_{j-1-k} \left( -d + \sigma \sqrt{T} \right) \varphi(d) \right]$$

From the martingale restriction, that is:

$$S_t = e^{-r_T} E_Q [S_T | S_t]$$

we can derive the first moment of the underlying asset log-return under the risk-neutral density. Indeed, equation (A.4.11.) implies that:

$$1 = e^{-r_T} E_Q \left[ \frac{S_T}{S_t} | S_t \right]$$

In order to evaluate expression (A.4.12.), we need to compute the following integral:

$$I_j^{***} = \int_{-\infty}^{+\infty} S_t e^{\mu_T + \sigma \sqrt{T}} H_j(z) \varphi(z) \, dz$$

for $j = [3, 4]$. Note that when the exercise price is equal to zero and the limit of integration are taken between minus and plus infinity, the integral $I_j^{***}$ is equivalent to the integral $I_j^*$. Thus, integrating expression (A.4.14.) by parts yields for $j = [3, 4]$:

$$I_j^{***} = (\sigma \sqrt{T})^j S_t \int_{-\infty}^{+\infty} e^{\mu_T + \sigma \sqrt{T}} \varphi(z) \, dz$$

Equation (A.4.12.) then becomes:

$$1 = e^{-r_T} \int_{-\infty}^{+\infty} \left( e^{\mu_T + \sigma \sqrt{T}} \right) \varphi(z) \, dz$$

Taking the logarithm of expression (A.4.15.) and rearranging terms, yields:

$$\mu_T = r_T - \frac{1}{2} \sigma^2 T - \ln \left[ 1 + \frac{\gamma_1 (F)}{3!} \sigma^3 \gamma_3/2 + \frac{\gamma_2 (F)}{4!} \sigma^4 \gamma^2 \right]$$
Substituting this expression into equation (A.4.1) and using Hermite polynomials such as $H_0(z) = 1, H_1(z) = z, H_2(z) = z^2 - 1$, the value of an European call becomes:

$$C_{CS} = \frac{S_t \Phi (d^*)}{1 + \frac{\gamma_1(f)}{4!} \sigma^3 \tau^{3/2} + \frac{\gamma_2(f)}{4!} \sigma^4 \tau^2} - e^{-r \tau} K \Phi (d^* - \sigma \sqrt{\tau}) \quad \text{(A.A.17.)}$$

$$+ \frac{\gamma_1(f) S_t}{3! \left( 1 + \frac{\gamma_1(f)}{3!} \sigma^3 \tau^{3/2} + \frac{\gamma_2(f)}{4!} \sigma^4 \tau^2 \right)} \left( \left( 2 \sigma^2 \tau - d^* \sigma \sqrt{\tau} \right) \varphi (d^*) + \sigma \sqrt{\tau} \right)$$

$$+ \sigma^3 \tau^{3/2} \Phi (d^*) \}$$

that is:

$$C_{CS} = \frac{S_t \Phi (d^*)}{1 + \frac{\gamma_1(f)}{4!} \sigma^3 \tau^{3/2} + \frac{\gamma_2(f)}{4!} \sigma^4 \tau^2} - e^{-r \tau} K \Phi (d^* - \sigma \sqrt{\tau})$$

$$+ \frac{\gamma_1(f) S_t}{3! \left( 1 + \frac{\gamma_1(f)}{3!} \sigma^3 \tau^{3/2} + \frac{\gamma_2(f)}{4!} \sigma^4 \tau^2 \right)} \left( \left( 2 \sigma^2 \tau - d^* \sigma \sqrt{\tau} \right) \varphi (d^*) + \sigma \sqrt{\tau} \right)$$

$$+ \sigma^3 \tau^{3/2} \Phi (d^*) \}$$

with:

$$d^* = \frac{\log (S_t / Ke^{-r \tau}) + r \tau + \frac{1}{2} \sigma^2 \tau - \ln \left( 1 + \frac{\gamma_1(f)}{3!} \sigma^3 \tau^{3/2} + \frac{\gamma_2(f)}{4!} \sigma^4 \tau^2 \right)}{\sigma \sqrt{\tau}}$$

Elimination of terms involving $\sigma^3 \tau^{3/2}$ and $\sigma^4 \tau^2$, which are small for options of same maturities, leads to:

$$C_{CS} = C_{BS} + \frac{\gamma_1(f)}{3!} S_t \sigma \sqrt{\tau} \left( 2 \sigma \sqrt{\tau} - d \right) \varphi (d) \quad \text{(A.A.18.)}$$

$$+ \frac{\gamma_2(f)}{4!} S_t \sigma \sqrt{\tau} \left( d^2 - 3d \sigma \sqrt{\tau} - 1 \right) \varphi (d)$$

where $d^* = d$ and $d$ is defined as in Black and Scholes (1973) formula.

### Appendix 5

Under the hypotheses of existence of the five first non-central moments of the underlying asset log-return density, the choice of the normal as the approximate density of the continuous compound return density, and perfection and
completeness of financial markets, the fair price of an European call $C_R$ written on a stock $S_t$ with strike price $K$ can be written as:

$$C_R = C_{BS} + \gamma_1(f) Q_3'' + \gamma_2(f) Q_4'' + \gamma_1(f)^2 Q_5''$$

(31)

where $Q_3''$ and $Q_4''$ are the same as in (29) and:

$$Q_5'' = \frac{10}{6!} S_t \sigma \sqrt{T} \left( d^4 - 6d^2 - 5d^3 \sigma \sqrt{T} + 15d \sigma \sqrt{T} + 3 \right) \varphi (d)$$

where terms involving three and higher order powers of $\sigma \sqrt{T}$ have been neglected.

**Proof.** Following the same approach as previously but using now a normal Egdeworth series expansion for the risk-neutral density of the underlying asset log-return with $H_6(z) = z^6 - 15z^4 + 45z^2 - 15$, yields:

$$C_R = S_t \Phi (d^{**}) - e^{-rT} K \Phi (d^{**} - \sigma \sqrt{T})$$

(A.5.1.)

$$+ \frac{\gamma_1(f)}{3!} S_t \varphi (d^{**}) \left( 2\sigma^2 T - d^{**} \sigma \sqrt{T} \right)$$

$$+ \frac{\gamma_2(f)}{4!} S_t \varphi (d^{**}) \left( 3 \sigma^3 T^{3/2} - 3d^{**} \sigma^2 T + d^{**^2} \sigma \sqrt{T} - \sigma \sqrt{T} \right)$$

$$+ \frac{10 \gamma_1(f)^2}{6!} S_t \varphi (d^{**}) \left[ \sigma \sqrt{T} \left( d^4 - 6d^2 + 3 \right) + \sigma^2 T \left( -5d^3 + 15d \right) + \sigma^3 T^{3/2} \left( 10d^2 - 10 \right) + -10d \sigma T^2 + 5\sigma^5 T^{5/2} \right]$$

with:

$$d^{**} = \frac{\log \left( S_t / K e^{-rT} \right) + rT + \frac{1}{2} \sigma^2 T - \ln \left[ 1 + \frac{\gamma_1(f)}{3!} \sigma^3 T^{3/2} + \frac{\gamma_2(f)}{4!} \sigma^4 T^2 + \frac{10 \gamma_1(f)^2}{6!} \sigma^6 T^{3/2} \right]}{\sigma \sqrt{T}}$$

Deleting terms involving three and higher orders of $\sigma \sqrt{T}$, which are small for options of same maturities, leads to:

$$C_R = C_{BS} + \frac{\gamma_1(f)}{3!} S_t \sigma \sqrt{T} \left( 2\sigma \sqrt{T} - d \right) \varphi (d)$$

(A.5.2.)

$$+ \frac{\gamma_2(f)}{4!} S_t \sigma \sqrt{T} \left( d^2 - 3d \sigma \sqrt{T} - 1 \right) \varphi (d) +$$

$$+ 10 \frac{\gamma_1(f)^2}{6!} S_t \sigma \sqrt{T} \left( d^4 - 6d^2 - 5d^3 \sigma \sqrt{T} + 15d \sigma \sqrt{T} + 3 \right) \varphi (d)$$

where $d^{**} = d$ and $d$ is defined as in Black and Scholes (1973) formula.

**Appendix 6**

39
For any European call, the following equality is verified:

\[ K \varphi \left[ d (\sqrt{T}) - \sigma \sqrt{T} \right] = S_t e^{rT} \varphi \left[ d (\sqrt{T}) \right] \quad (A.4.9.) \]

**Proof.** Following Stoll and Whaley (1993, p.245), we can define \( d \) in a general setting such as:

\[ d = \frac{\log (S_t/K) + \mu \tau + \sigma^2 \tau}{\sigma \sqrt{T}} \quad (A.6.1.) \]

where \( \mu \) is not restricted.

Developing the square of \( d \) gives:

\[ (d - \sigma \sqrt{T})^2 = d^2 - 2d \sigma \sqrt{T} + \sigma^2 \tau \quad (A.6.2.) \]

\[ = d^2 - 2 \ln (S_t/K) + \mu \tau + \sigma^2 \tau + \sigma^2 \tau \]

\[ = d^2 - 2 \ln (S_t/K) + \mu \tau + \frac{\sigma^2 \tau}{2} \]

\[ = d^2 - 2 \ln \left( S_t e^{\mu \tau + \sigma^2 \tau/2} / K \right) \]

Evaluating the standard normal density at \( (d - \sigma \sqrt{T}) \), we get:

\[ \varphi \left( d - \sigma \sqrt{T} \right) = \frac{1}{\sqrt{2\pi}} e^{-\left(d - \sigma \sqrt{T}\right)^2/2} \quad (A.6.3.) \]

\[ = \frac{1}{\sqrt{2\pi}} e^{-d^2/2 + \ln \left( S_t e^{\mu \tau + \sigma^2 \tau/2} / K \right)} \]

\[ = \frac{1}{\sqrt{2\pi}} e^{-d^2/2 + \ln \left( S_t e^{\mu \tau + \sigma^2 \tau/2} / K \right)} \]

\[ = \varphi \left( d \right) S_t e^{\mu \tau + \sigma^2 \tau/2} / K \]

Rearranging equation (A.6.3.), we obtain the following identity:

\[ K \varphi \left( d - \sigma \sqrt{T} \right) = S_t e^{\mu \tau + \sigma^2 \tau/2} \varphi \left( d \right) \quad (A.6.4.) \]

In the particular case where \( \mu \tau = r \tau - \sigma^2 \tau/2 \) (see Black and Scholes, 1973), this expression becomes:

\[ K \varphi \left[ d (\sqrt{T}) - \sigma \sqrt{T} \right] = S_t e^{r \tau} \varphi \left[ d (\sqrt{T}) \right] \quad (A.6.5.) \]

**Appendix 7**
When the European call market price is given by the Jarrow and Rudd (1982) formula, the implied volatility function can be written such as:

\[
ISD_{JR} = \Psi [S_t, K, \tau, \sigma_{GC}, f, l, S_T, \kappa_3, \kappa_4]
\]

(35)

\[
\simeq \sigma \sqrt{\tau} + \left\{ \frac{\gamma_1(f) - \gamma_1(l)}{3!} \right\} (S_t e^{\tau} e^{2\sigma^2\tau} - 1)^{3/2} \\
\times \left\{ \frac{l(K)}{K \sigma \sqrt{\tau} \varphi(d)} \right\} \\
+ \left\{ \frac{\gamma_2(f) - \gamma_2(l)}{4!} \right\} (S_t e^{\tau} e^{\sigma^2\tau} - 1)^2 \\
\times \left\{ \frac{l(K)}{K \sigma \sqrt{\tau} \varphi(d)} \right\}
\]

Proof. The implied volatility function \( ISD \) corresponds to the volatility \( \sigma \) that equates the market price of the option to the value of the Black and Scholes (1973) formula, given values of other parameters fixed, that is:

\[
C = e^{-rT} \int_{S_T = K}^{S_T = \infty} (S_T - K) f(S_T) dS_T
\]

(A.7.1.)

\[
= S_t \Phi [d (\Psi)] - Ke^{-rT} \Phi [d (\Psi) - \Psi]
\]

where \( d (\Psi) = [\log(S_t/Ke^{-rT}) + 0.5\Psi^2] \Psi^{-1} \) and \( \Phi(.) \) represent respectively the Black and Scholes’ measure of moneyness evaluated at the implied volatility level.

A linear approximation of this expression around the “true” volatility of the underlying asset \( \sigma \sqrt{\tau} \) gives:

\[
C \cong S_t \Phi [d (\sigma \sqrt{\tau})] - Ke^{-rT} \Phi [d (\sigma \sqrt{\tau}) - \sigma \sqrt{\tau}]
\]

\[
+ S_t \varphi [d (\sigma \sqrt{\tau}) \left( -\frac{\ln S_t/Ke^{-rT}}{\sigma^2\tau} + \frac{1}{2} \right) (\Psi - \sigma \sqrt{\tau})]
\]

(A.7.2.)

\[
-Ke^{-rT} \varphi [d (\sigma \sqrt{\tau}) - \sigma \sqrt{\tau}] \left( -\frac{\ln S_t/Ke^{-rT}}{\sigma^2\tau} - \frac{1}{2} \right) (\Psi - \sigma \sqrt{\tau})
\]

with \( \varphi(.) \) the standard normal density function.

Using the following equality (see Appendix 6):

\[
K \varphi [d (\sigma \sqrt{\tau}) - \sigma \sqrt{\tau}] = S_t e^{\tau} \varphi [d (\sigma \sqrt{\tau})]
\]

(A.7.3.)

equation (A.7.2.) simplifies to:

\[
C \cong S_t \Phi [d (\sigma \sqrt{\tau})] - Ke^{-rT} \Phi [d (\sigma \sqrt{\tau}) - \sigma \sqrt{\tau}]
\]

(A.7.4.)

(35)

+ S_t \psi [d (\sigma \sqrt{\tau})] (\Psi - \sigma \sqrt{\tau})

If the risk-neutral density is obtained by a log-normal series expansion, substituting the Jarrow-Rudd formula for the market price in the expression (A.7.4.) and rearranging terms give the desired result.
If we use instead a Gram-Charlier type A or an Edgeworth series expansion for the risk-neutral pricing density, equating (A.7.4.) with the Corrado-Su (1996-b and 1997-b) formula or the Rubinstein (1998) formula leads to expression (36) or (37).

Appendix 8

When the European call market price is given by the Jarrow-Rudd (1982) formula, the Greek parameters of the call can be written respectively as equation (38), (39), (40), (41) and (42).

Proof: Consider the Jarrow-Rudd (1982) formula of an European call option:

\[ C_{JR} \simeq C_{BS} + \lambda_1 Q_3 + \lambda_2 Q_4 \]  (A.8.1.)

with:

\[ \begin{cases} 
\lambda_1 = [\gamma_1 (f) - \gamma_1 (l)] \\
\lambda_2 = [\gamma_2 (f) - \gamma_2 (l)]
\end{cases} \]

and:

\[ \begin{cases} 
Q_3 = - (S_t e^{r\tau})^3 (e^{\sigma^2 \tau} - 1)^{3/2} e^{-r\tau} (d - 2 \sigma \sqrt{\tau}) \frac{l(K)}{K \sigma \sqrt{\tau}} \\
Q_4 = (S_t e^{r\tau})^4 (e^{\sigma^2 \tau} - 1)^{2} e^{-r\tau} (d^2 - 5d \sigma \sqrt{\tau} + 6 \sigma^2 \tau - 1) \frac{l(K)}{K \sigma \sqrt{\tau}}
\end{cases} \]

Differentiating the Jarrow-Rudd formula (A.8.1.) with respect to the underlying price, we get:

\[ \frac{\partial C_{JR}}{\partial S_t} = \frac{\partial C_{BS}}{\partial S_t} + \lambda_1 \frac{\partial Q_3}{\partial S_t} + \lambda_2 \frac{\partial Q_4}{\partial S_t} \]  (A.8.2.)

with:

\[ \frac{\partial Q_3}{\partial S_t} = - \frac{(e^{r\tau})^2}{3K \sigma \sqrt{\tau}} (e^{\sigma^2 \tau} - 1)^{3/2} \left[ 3 (S_t)^2 (d - 2 \sigma \sqrt{\tau}) l(K) \right. + (S_t)^3 \frac{\partial d}{\partial S_t} l(K) + (S_t)^3 (d - 2 \sigma \sqrt{\tau}) \frac{\partial l(K)}{\partial S_t} \left. \right] \]

\[ = -(S_t e^{r\tau})^2 \left( e^{\sigma^2 \tau} - 1 \right)^{3/2} \frac{l(K)}{3K \sigma \sqrt{\tau}} \left( -d^2 + 6d \sigma \sqrt{\tau} - 8\sigma^2 \tau + 1 \right) \]

and:

\[ \frac{\partial Q_4}{\partial S_t} = \frac{(e^{r\tau})^3}{4K^2 \sigma^3 \tau} \left( e^{\sigma^2 \tau} - 1 \right)^2 \left[ 4 (S_t)^3 (d^2 - 5d \sigma \sqrt{\tau}) l(K) + 6 \sigma^2 \tau - 1 \right] \frac{\partial d}{\partial S_t} - 5 \frac{\partial d}{\partial S_t} \frac{\partial \left( d \sigma \sqrt{\tau} \right)}{\partial S_t} \frac{\partial l(K)}{\partial S_t} \]

\[ = (S_t e^{r\tau})^3 \left( e^{\sigma^2 \tau} - 1 \right)^2 \frac{l(K)}{4K^2 \sigma^3 \tau^{3/2}} \left( -d^3 + 10d^2 \sigma \sqrt{\tau} \right. + 31d \sigma^2 \tau + 3d + 30\sigma^3 \tau^{3/2} - 10 \sigma \sqrt{\tau} \left. \right) \]
where:
\[
\begin{align*}
\frac{\partial d}{\partial S_t} &= \frac{1}{S_t \sigma \sqrt{T}} \\
\frac{\partial l(K)}{\partial S_t} &= -\frac{(d - \sigma \sqrt{T})}{S_t \sigma \sqrt{T}} l(K)
\end{align*}
\] (A.8.5.)

Substituting expression (A.8.3.) and (A.8.4.) in equation (A.8.1.) leads to the Delta formula (38) for the Jarrow and Rudd (1982) model.

Differentiating once again expression (A.8.1.) with respect to the underlying asset price, we have:
\[
\frac{\partial \Delta_{\text{JR}}}{\partial S_t} = \frac{\partial \Delta_{BS}}{\partial S_t} + \lambda_1 \left[ \frac{\partial Q''_{3}}{\partial S_t} (-d^2 + 6d \sigma \sqrt{T} - 8\sigma^2 \tau + 1) + Q_3'' \left( -2d \frac{\partial d}{\partial S_t} + 6 \frac{\partial d}{\partial S_t} \sigma \sqrt{T} \right) \right]
\]
\[+ \lambda_2 \left[ \frac{\partial Q''_{4}}{\partial S_t} (-d^3 + 10d^2 \sigma \sqrt{T} - 31d \sigma^2 \tau + 3d \right.
\]
\[+ 30\sigma^3 \tau^{3/2} - 10\sigma \sqrt{T} \left. \right) + Q_4'' \left( -3d^2 \frac{\partial d}{\partial S_t} + 20d \frac{\partial d}{\partial S_t} - 31 \frac{\partial d}{\partial S_t} \sigma^2 \tau + 3 \frac{\partial d}{\partial S_t} \right) \]
\] (A.8.6.)

with
\[
\frac{\partial Q''_{3}}{\partial S_t} = -\frac{(e^{\tau \gamma})^2}{3K \sigma \sqrt{T}} \left( e^{\sigma^2 \tau - 1} \right)^{3/2} \left[ 2S_t l(K) + (S_t)^3 \frac{\partial l(K)}{\partial S_t} \right]
\] (A.8.7.)
and:
\[
\frac{\partial Q''_{4}}{\partial S_t} = -\frac{(e^{\tau \gamma})^3}{4K^2 \sigma^3 \tau^{3/2}} \left( e^{\sigma^2 \tau - 1} \right)^2 \left[ 3S_t^2 l(K) + (S_t)^3 \frac{\partial l(K)}{\partial S_t} \right]
\] (A.8.8.)

Substituting expression (A.8.7.) and (A.8.8.) in equation (A.8.6.), factoring out \((S_t \sigma \sqrt{T})^{-1}\) leads to the Gamma formula (39) for the Jarrow and Rudd (1982) model.

Differentiating the Jarrow and Rudd equation (A.8.1.) with respect to the volatility gives:
\[
\frac{\partial C_{JR}}{\partial \sigma} = \frac{\partial C_{BS}}{\partial \sigma} + \lambda_1 \frac{\partial Q_3}{\partial \sigma} + \lambda_2 \frac{\partial Q_4}{\partial \sigma} - \frac{\partial \gamma_1}{\partial \sigma} Q_3 - \frac{\partial \gamma_2}{\partial \sigma} Q_4
\] (A.8.9.)
with:
\[
\frac{\partial Q_3}{\partial \sigma} = -(S_t e^{\sigma \tau})^3 \frac{e^{\sigma \tau}}{3!K} \left[ 3\sigma e^{\sigma \tau} \left( e^{\sigma \tau} - 1 \right)^{1/2} (d - 2\sigma \sqrt{\tau}) \right] \\
\times \frac{l(K)}{\sigma \sqrt{\tau}} + \left( e^{\sigma \tau} - 1 \right)^{3/2} \left( \frac{\partial l}{\partial \sigma} - 2\sqrt{\tau} \right) \frac{l(K)}{\sigma \sqrt{\tau}} \\
+ \left( \frac{\partial l}{\partial \sigma} - 2\sqrt{\tau} \right)^2 \frac{l(K)}{\sigma \sqrt{\tau}} \left( \sigma \sqrt{\tau} \right)^{-2} \\
= -(S_t e^{\sigma \tau})^3 \frac{e^{\sigma \tau}}{3!} \left( e^{\sigma \tau} - 1 \right)^{1/2} \frac{l(K)}{K \sigma^2 \sqrt{\tau}} \left[ 3\sigma e^{\sigma \tau} \\
\times (d^2 - 3d^2 \sigma \sqrt{\tau} + 3\sigma \sqrt{\tau} - 3d + 2d\sigma^2 \tau) \right]
\]

Similarly:
\[
\frac{\partial Q_4}{\partial \sigma} = -(S_t e^{\sigma \tau})^4 \frac{e^{\sigma \tau}}{4!K^2} \left[ 4\sigma e^{\sigma \tau} \left( e^{\sigma \tau} - 1 \right) \right] \left( d^2 - 5d \sigma \sqrt{\tau} + 6\sigma^2 \tau - 1 \right) \left( d^2 - 5d \sigma \sqrt{\tau} + 6\sigma^2 \tau - 1 \right) \\
\times \frac{l(K)}{\sigma^2 \sqrt{\tau}} + \left( e^{\sigma \tau} - 1 \right)^2 \left( \frac{\partial l}{\partial \sigma} - 2\sigma \frac{l(K)}{\sigma \sqrt{\tau}} \left( \sigma \sqrt{\tau} \right)^{-2} \right) \\
= -(S_t e^{\sigma \tau})^4 \frac{e^{\sigma \tau}}{4!K^2} \frac{l(K)}{\sigma^2 \sqrt{\tau}} \left[ 4\sigma e^{\sigma \tau} \left( e^{\sigma \tau} - 1 \right) \right] \\
\times (d^2 - 5d \sigma \sqrt{\tau} + 6\sigma^2 \tau - 1) + \left( e^{\sigma \tau} - 1 \right)^2 \\
\times (-2d^2 - 2d \sigma \sqrt{\tau} + 7\sigma^2 \tau) + \left( e^{\sigma \tau} - 1 \right)^2 \\
(d^2 - 5d \sigma \sqrt{\tau} + 6\sigma^2 \tau - 1) \left( d^2 - d\sigma \sqrt{\tau} - 3 \right)
\]

and:
\[
\begin{align*}
\frac{\partial \gamma_1(l)}{\partial \sigma} &= 3\sigma \tau e^{\sigma \tau} \left[ \left( e^{\sigma \tau} - 1 \right)^{-\frac{1}{2}} + \left( e^{\sigma \tau} - 1 \right)^{\frac{1}{2}} \right] \\
\frac{\partial \gamma_2(l)}{\partial \sigma} &= 4\sigma \tau e^{\sigma \tau} \left[ 8 + 15 \left( e^{\sigma \tau} - 1 \right) \\
&\quad + 9 \left( e^{\sigma \tau} - 1 \right)^2 + 2 \left( e^{\sigma \tau} - 1 \right)^3 \right]
\end{align*}
\]
where:

\[
\begin{align*}
\frac{\partial_b}{\partial(K)} &= -\frac{(d-\sigma\sqrt{\tau})}{\sigma} \\
\frac{\partial_b}{\partial(\sigma\tau)} &= \frac{(\sigma^2-\sigma\tau-1)}{\sigma}l(K)
\end{align*}
\]  

(A.8.13.)

Substituting expression (A.8.11.) and (A.8.12.) in equation (A.8.9.), factoring out \(S_t\sqrt{\tau}\) leads to the Vega formula (39) for the Jarrow-Rudd (1982) model.

Differentiating expression (A.8.1.) with respect to the excess of skewness and to the excess of the excess kurtosis leads directly to the equation (41) and (42) of the Jarrow-Rudd Khi and Psi, that is:

\[
\begin{align*}
\frac{\partial C_{JR}}{\partial(\gamma_1)} &= Q_3^C \\
\frac{\partial C_{JR}}{\partial(\gamma_2)} &= \Psi_{JR}^C Q_4
\end{align*}
\]  

(A.8.14.)

Appendix 9

When the European call market price is given by the Corrado and Su (1996-b and 1997-b) formula, the Greeks of a call can be written respectively such as equations (43), (44), (45), (46) and (47).

**Proof**: Consider the Corrado and Su (1996-b and 1997-b) formula of an European call option:

\[
C_{CS} = C_{BS} + \gamma_1(f) Q_3^C + \gamma_2(f) Q_4^C
\]  

(A.9.1.)

with:

\[
\begin{align*}
Q_3^C &= \frac{1}{3!} S_t \sigma \sqrt{\tau} (2\sigma \sqrt{\tau} - d) \varphi(d) \\
Q_4^C &= \frac{1}{4!} S_t \sigma \sqrt{\tau} (d^2 - 3d \sigma \sqrt{\tau} - 1) \varphi(d)
\end{align*}
\]

Differentiating the Corrado and Su formula (A.9.1.) with respect to the underlying asset price, we get:

\[
\frac{\partial C_{CS}}{\partial S_t} = \frac{\partial C_{BS}}{\partial S_t} + \gamma_1(f) \frac{\partial Q_3^C}{\partial S_t} + \gamma_2(f) \frac{\partial Q_4^C}{\partial S_t}
\]  

(A.9.2.)

with:

\[
\frac{\partial Q_3^C}{\partial S_t} = \frac{1}{3!} \left[ \sigma \sqrt{\tau} (2\sigma \sqrt{\tau} - d) \varphi(d) - S_t \frac{\partial d}{\partial S_t} \varphi(d) \\
+ S_t (2\sigma \sqrt{\tau} - d) \frac{\partial \varphi(d)}{\partial d} \frac{\partial d}{\partial S_t} \right]
\]  

(A.9.3.)

\[
= \frac{1}{3!} \varphi(d) (d^2 - 3d \sigma \sqrt{\tau} + 2\sigma^2 \tau - 1)
\]

45
and:

\[
\frac{\partial Q_4'}{\partial S_t} = \frac{1}{4!} \sigma \sqrt{\tau} \left[ (d^2 - 3d \sigma \sqrt{\tau} - 1) \varphi(d) \right. \\
+ S_t \left( 2d \frac{\partial d}{\partial S_t} - 3 \frac{\partial d}{\partial S_t} \sigma \sqrt{\tau} \right) \varphi(d) \\
+ S_t \left( d^2 - 3d \sigma \sqrt{\tau} - 1 \right) \frac{\partial \varphi(d)}{\partial d} \frac{\partial d}{\partial S_t} \left] \right.
\]

\(= \frac{1}{4!} \varphi(d) \left( -d^3 + 4d^2 \sigma \sqrt{\tau} + 3d - d \sigma^2 \tau - 4 \sigma \sqrt{\tau} \right) \)  (A.9.4.)

where:

\(\frac{\partial \varphi(d)}{\partial d} = -d \varphi(d)\)

Substituting expression (A.9.3.) and (A.9.4.) in equation (A.9.1), factoring out \(\varphi(d)\) leads to the Delta formula (43) for the Corrado-Su (1996-b and 1997-b) model.

Differentiating once again expression (A.9.1) with respect to the underlying asset price, we have:

\[
\frac{\partial \Delta_{CS}}{\partial S_t} = \frac{\partial \Delta_{BS}}{\partial S_t} + \gamma_1(f) \frac{\partial^2 Q_3'}{\partial S_t^2} + \gamma_2(f) \frac{\partial^2 Q_4'}{\partial S_t^2} \]  (A.9.5.)

where:

\[
\frac{\partial^2 Q_3'}{\partial S_t^2} = \frac{1}{3!} \left[ \frac{\partial \varphi(d)}{\partial d} \frac{\partial d}{\partial S_t} \left( d^2 - 3d \sigma \sqrt{\tau} + 2 \sigma^2 \tau - 1 \right) \right. \\
+ \varphi(d) \left( 2d \frac{\partial d}{\partial S_t} - 3 \frac{\partial d}{\partial S_t} \sigma \sqrt{\tau} \right) \] 

\(= \frac{1}{3!} \frac{\varphi(d)}{S_t \sigma \sqrt{\tau}} \left[-d^3 + 3d^2 \sigma \sqrt{\tau} - 2d \sigma^2 \tau + 3d - 3 \sigma \sqrt{\tau} \right] \)  (A.9.6.)

and:

\[
\frac{\partial^2 Q_4'}{\partial S_t^2} = \frac{1}{4!} \left[ \frac{\partial \varphi(d)}{\partial d} \frac{\partial d}{\partial S_t} \left( -d^3 + 4d^2 \sigma \sqrt{\tau} + 3d - d \sigma^2 \tau - 4 \sigma \sqrt{\tau} \right) \right. \\
+ \varphi(d) \left( -3d^2 \frac{\partial d}{\partial S_t} + 8d \frac{\partial d}{\partial S_t} \sigma \sqrt{\tau} + 3 \frac{\partial d}{\partial S_t} - \frac{\partial d}{\partial S_t} \sigma^2 \tau \right) \] 

\(= \frac{1}{4!} \frac{\varphi(d)}{S_t \sigma \sqrt{\tau}} \left( d^4 - 4d^3 \sigma \sqrt{\tau} + 3d^2 \sigma^2 \tau - 6d^2 + 12d \sigma \sqrt{\tau} \right. \right. \\
-3 \sigma^2 \tau + 3 \)  (A.9.7.)

Substituting expression (A.9.6.) and (A.9.7.) in equation (A.9.5.), factoring out \(\varphi(d) (S_t \sigma \sqrt{\tau})^{-1} \) leads to the Gamma formula (44) for the Corrado-Su (1996-b and 1997-b) model.
Differentiating the Corrado-Su equation (A.9.1.) with respect to the volatility, we get:

\[
\frac{\partial C_{CS}}{\partial \sigma} = \frac{\partial C_{BS}}{\partial \sigma} + \gamma_1 (f) \frac{\partial Q'_3}{\partial \sigma} + \gamma_2 (f) \frac{\partial Q'_4}{\partial \sigma}
\]  

(A.9.8.)

where:

\[
\frac{\partial Q'_3}{\partial \sigma} = \frac{1}{3!} S_t \left[ \left(4\sigma \tau - \frac{\partial d}{\partial \sigma} \sigma \sqrt{\tau} - d\sqrt{\tau}\right) \varphi (d) \right. \\
\left. + (2\sigma^2 \tau - d\sigma \sqrt{\tau}) \frac{\partial \varphi (d)}{\partial d} \frac{\partial d}{\partial \sigma} \right] \\
= \frac{1}{3!} \varphi (d) S_t \sqrt{\tau} (-d^3 + 3d^2 \sigma \sqrt{\tau} - 2d \sigma^2 \tau + 3d - 3 \sigma \sqrt{\tau})
\]  

(A.9.9.)

and:

\[
\frac{\partial Q'_4}{\partial \sigma} = \frac{1}{4!} S_t \left[ \left(2d \frac{\partial d}{\partial \sigma} \sigma \sqrt{\tau} + d^2 \frac{\partial d}{\partial \sigma} \sigma \sqrt{\tau} - 3\frac{\partial d}{\partial \sigma} \sigma \sqrt{\tau} - 6d \sigma \tau \right) \varphi (d) \right. \\
\left. + (d^2 \sigma \sqrt{\tau} - 3d \sigma^2 \tau - \sigma \sqrt{\tau}) \frac{\partial \varphi (d)}{\partial d} \frac{\partial d}{\partial \sigma} \right] \\
= \frac{1}{4!} \varphi (d) S_t \sqrt{\tau} (d^4 - 4d^3 \sigma \sqrt{\tau} + 3d^2 \sigma^2 \tau - 2d^2 - 3 \sigma^2 \tau - 1)
\]  

(A.9.10.)

where:

\[
\frac{\partial \varphi (d)}{\partial d} = -d \varphi (d)
\]  

(A.9.11.)

Substituting expression (A.9.10.) and (A.9.11.) in equation (A.9.8.), factoring out \( \varphi (d) S_t \sqrt{\tau} \) leads to the Vega formula (45) for the Corrado-Su (1996-b and 1997-b) model.

Differentiating expression (A.9.1.) with respect to the skewness and to the excess kurtosis leads directly to the equations (45) and (46) of the Corrado-Su Khi and Psi, that is:

\[
\begin{aligned}
\frac{\partial C_{CS}}{\partial \gamma_3 (f)} &= \chi_{CS}^C = Q_3' \\
\frac{\partial C_{CS}}{\partial \gamma_4 (f)} &= \chi_{CS}^C = Q_4'
\end{aligned}
\]  

(A.9.12.)

**Appendix 10**

When the European call market price is given by the Rubinstein (1998) formula, the Greeks of a call can be written respectively such as equations (48), (49), (50), (51) and (52).

**Proof:** Consider the Rubinstein (1998) formula of an European call option:

\[
C_R = C_{BS} + C_{CS} + \gamma_1 (f)^2 Q_5'
\]  

(A.10.1.)

with:

\[
Q_5' = \frac{10}{6!} S_t \sigma \sqrt{\tau} (d^4 - 6d^2 - 5d^3 \sigma \sqrt{\tau} + 15d \sigma \sqrt{\tau} + 3) \varphi (d)
\]
Differentiating this expression with respect to the underlying price, we have:

\[
\frac{\partial C_R}{\partial S_t} = \frac{\partial C_{BS}}{\partial S_t} + \frac{\partial C_{CS}}{\partial S_t} + \gamma_1 (f)^2 \frac{\partial Q_5''}{\partial S_t} \tag{A.10.2.}
\]

where \( \frac{\partial C_{BS}}{\partial S_t} \) and \( \frac{\partial C_{CS}}{\partial S_t} \) are defined as previously, and:

\[
\frac{\partial Q_5''}{\partial S_t} = \frac{10}{6!} \sqrt{\tau} \left[(d^4 - 5d^3 \sigma \sqrt{\tau} - 6d^2 + 15d \sigma \sqrt{\tau} + 3) \varphi (d) \right. \tag{A.10.3.}
\]

\[
+ S_t \left(4d^5 \frac{\partial d}{\partial S_t} - 15d^2 \frac{\partial d}{\partial S_t} - 12d \frac{\partial d}{\partial S_t} + 15 \frac{\partial d}{\partial S_t} \sigma \sqrt{\tau}\right) \times \varphi (d) + S_t \left(d^4 - 5d^3 \sigma \sqrt{\tau} - 6d^2 + 15d \sigma \sqrt{\tau} + 3\right) \frac{\partial \varphi (d)}{\partial d} \frac{\partial d}{\partial S_t} \bigg] \tag{A.10.4.}
\]

Substituting this expression in equation (A.10.1) and factoring out \( \varphi (d) \) leads to the Delta formula (48) for the Rubinstein (1998) model.

Differentiating expression (A.10.1.) with respect to the underlying asset price, we have:

\[
\frac{\partial \Delta_{CS}}{\partial S_t} = \frac{\partial \Delta_{BS}}{\partial S_t} + \frac{\partial \Delta_{CS}}{\partial S_t} + \gamma_1 (f)^2 \frac{\partial^2 Q_5''}{\partial S_t^2} \tag{A.10.4.}
\]

where \( \frac{\partial \Delta_{BS}}{\partial S_t} \) and \( \frac{\partial \Delta_{CS}}{\partial S_t} \) are defined as previously, and:

\[
\frac{\partial^2 Q_5''}{\partial S_t^2} = \frac{10}{6!} \left[ \frac{\partial \varphi (d)}{\partial d} \frac{\partial d}{\partial S_t} \left(-d^5 + 6d^4 \sigma \sqrt{\tau} - 5d^3 \sigma^2 \tau + 10d^3\right) \right. \tag{A.10.5.}
\]

\[
-36d^2 \sigma \sqrt{\tau} + 15d \sigma^2 \tau
\]

\[
-15d15d^2 \sigma^2 \tau - 15d + 18\sigma \sqrt{\tau}
\]

\[
+ \varphi (d) \left(-5d^4 \frac{\partial d}{\partial S_t} + 24d^3 \frac{\partial d}{\partial S_t} - 15d^2 \frac{\partial d}{\partial S_t} \sigma^2 \tau \right.
\]

\[
+30d^2 \frac{\partial d}{\partial S_t} - 72d \frac{\partial d}{\partial S_t} - 15 \frac{\partial d}{\partial S_t} \bigg] \tag{A.10.5.}
\]

Substituting this expression in equation (A.10.4.) and factoring out \( \varphi (d) \) \((S_t \sigma \sqrt{\tau})^{-1}\) leads to the Gamma formula (49) for the Rubinstein (1998) model.
Differentiating the Rubinstein’s equation (A.10.1.) with respect to the volatility gives:

$$\frac{\partial C_R}{\partial \sigma} = \frac{\partial C_{BS}}{\partial \sigma} + \frac{\partial C_{CS}}{\partial \sigma} + \gamma_1 (f)^2 \frac{\partial Q_5^*}{\partial \sigma}$$  \hspace{1cm} (A.10.6.)

where $\frac{\partial C_{BS}}{\partial \sigma}$ and $\frac{\partial C_{CS}}{\partial \sigma}$ are defined as previously, and:

$$\frac{\partial Q_5^*}{\partial \sigma} = \frac{10}{6!} S_t \left[ \left( 4d^3 \frac{\partial d}{\partial \sigma} \sigma \sqrt{\tau} + d^4 \sqrt{\tau} - 12d \frac{\partial d}{\partial \sigma} \sigma \sqrt{\tau} \right) - 6d^2 \sqrt{\tau} - 15d^2 \frac{\partial d}{\partial \sigma} \sigma^2 \tau - 10d^3 \sigma \tau \\
+ 15 \frac{\partial d}{\partial \sigma} \sigma^2 \tau + 30d \sigma \tau + 3 \sqrt{\tau} \right) \varphi(d) \\
+ \left( d^4 \sigma \sqrt{\tau} - 6d^2 \sigma \sqrt{\tau} - 5d^3 \sigma^2 \tau \\
+ 15d \sigma^2 \tau + 3 \sigma \sqrt{\tau} \right) \frac{\partial \varphi(d)}{\partial d} \frac{\partial d}{\partial \sigma} \right]$$

$$= \frac{1}{3!} \varphi(d) S_t \sqrt{\tau} \left( d^6 - 6d^5 \sigma \sqrt{\tau} + 5d^4 \sigma^2 \tau - 9d^2 \\
+ 30d^3 \sigma \sqrt{\tau} - 30d^2 \sigma^2 \tau + 39d^3 + 15 \sigma \sqrt{\tau} \right)$$

Substituting expression (A.10.7.) in equation (A.10.6.), factoring out $\varphi(d) S_t \sqrt{\tau}$ leads to the Vega formula (50) for the Rubinstein (1998) model.

Differentiating expression (A.10.1.) with respect to the skewness and to the excess kurtosis leads directly to the Rubinstein’s Khi and Psi (equations (51) and (52)), that is:

$$\begin{align*}
\frac{\partial C_R}{\partial \delta} &= \chi \frac{\partial C_{CS}}{\partial \delta} + 2 \gamma_1 (f) Q_5^* \\
\frac{\partial C_R}{\partial \gamma_1(f)} &= \Psi \frac{\partial C_{CS}}{\partial \gamma_1(f)}
\end{align*}$$

(A.10.8.)
Appendix 11

All the following Figures result from simulations considering an at-the-money price of 2250 (FRF), an annualized implied volatility of 23%, an annualized skewness parameter of -.7, an annualized kurtosis index of 3.53 and an annualized risk-free rate of 3.41%. These values happen to be approximately the mean values considering market data and backed-out parameters for the Jarrow-Rudd (1982) model on the French market on the sample 01/1997-12/1998 for Long Term CAC 40 options (see Capelle-Blancard et al., 2001, for details). The considered option maturity is three months and the moneyness varies from -.35% to .35%; with these different values, simulated option prices range from 7 to 511(FRF) according to the various models and parameters.

Figure 1: Sensitivities of Option Price to the Excess Moments for the Jarrow-Rudd (1982) Model

- \((-Q_3)\) and \(Q_4\) -

Figure 2: Sensitivities of Option Price to the Excess Moments for the Corrado-Su (1996-c and 1997-c) Model

- \(Q_3'\) and \(Q_4'\) -
Figure 3: Sensitivities of Option Price to the Excess Moments for the Rubinstein (1998) Model – $Q_3^*$, $Q_4^*$ and $Q_5^*$

Figure 4: Impacts of Higher Moments on Option Prices
Figure 5: Implied Density Functions

Figure 6: Implied Volatility Smile Functions
Figure 7: Effects of Excess Skewness on Implied Volatility Functions

Figure 8: Effects of Excess Kurtosis on Implied Volatility Functions
Figure 9: The Deltas

Figure 10: The Gammas

Note that no significant difference can be highlighted between Black-Scholes and Jarrow-Rudd models in term of Deltas for the considered values of the simulation.

This simulation exhibits some degenerated negative – but low – values of Gamma for the considered values of the simulation.
For practical reasons, Jarrow-Rudd Psi parameter has been multiplied by $10^6$ and those of Rubinstein by $10^5$. 
For practical reasons, Jarrow-Rudd Khi parameter has been multiplied by $10^9$ and those of Rubinstein by $10^9$.

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51 For practical reasons, Jarrow-Rudd Khi parameter has been multiplied by $10^9$ and those of Rubinstein by $10^9$. 